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AN ELEMENTARY TREATISE

ON

THE INTEGRAL CALCULUS.

BY THE SAME AUTHOR.

AN ELEMENTARY TREATISE
ON
THE DIFFERENTIAL CALCULUS.
Second Edition.
LONDON: LONGMANS, GREEN, AND CO.

AN ELEMENTARY TREATISE

ON

THE INTEGRAL CALCULUS,

CONTAINING

APPLICATIONS TO PLANE CURVES
AND SURFACES;

WITH

NUMEROUS EXAMPLES.

BY

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A·D·1591

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CORRIGENDA.

Page 99, line 4, *for* Arts. 48, 59, *read* Arts. 47, 48.
—— 116, last line, *for* 1·2·3 . . . n, *read* 1·2·3 . . . $(n-1)$.
—— 217, last line, *for* $F_1\ F_1$, *read* $F,\ F_1$.

INTEGRAL CALCULUS.

CHAPTER I.

ELEMENTARY FORMS OF INTEGRATION.

1. **Integration.**—The Integral Calculus is the inverse of the Differential. In the more simple case to which this treatise is limited, the object of the Integral Calculus is to *find a function of a single variable when its differential is known.*

Let the differential be represented by $F(x)\,dx$, then the function whose differential is $F(x)\,dx$ is called its *integral*, and is represented by the notation

$$\int F(x)\,dx.$$

Thus, since in the notation of the Differential Calculus we have

$$df(x) = f'(x)\,dx,$$

the integral of $f'(x)\,dx$ is denoted by $f(x)$; or

$$\int f'(x)\,dx = f(x).$$

Moreover, as $f(x)$ and $f(x) + C$ (where C is any arbitrary quantity that does not vary with x) have the same differential, it follows, that to find the *general* form of the integral of $f'(x)\,dx$ it is necessary to add an arbitrary constant to $f(x)$; hence, we obtain, as the general expression for the integral in question

$$\int f'(x)\,dx = f(x) + C. \qquad (1)$$

In the subsequent integrals the constant C will be omitted, as it can always be supplied when necessary. In the applications of the Integral Calculus the value of the constant is determined in each case by the *data* of the problem, as will be more fully explained in a subsequent place.

The process of finding the *primitive* function or the *integral* of any given differential is called *integration.*

The expression $F(x)\,dx$ under the sign of integration is called an *element* of the integral; it is also, in the limit, the increment of the primitive function, when x is changed into $x + dx$ (Dif. Cal., Art. 7); accordingly, the process of integration may be regarded as the finding the *sum*[*] of an infinite number of such elements.

We shall consider the subject of Integration from this point of view in a subsequent chapter, but for the present shall regard Integration as being the inverse of Differentiation.

2. **Elementary Integrals.**—The smallest acquaintance with the Differential Calculus will at once suggest the integrals of many differentials. We commence with the simplest cases, *an arbitrary constant being in all cases understood.*

On referring to the elementary forms of differentiation established in Chapter I. Dif. Cal., we may write down at once the following integrals :—

$$\int x^m dx = \frac{x^{m+1}}{m+1}. \qquad (a)$$

$$\int \frac{dx}{x} = \log(x). \qquad (b)$$

$$\int \sin mx\,dx = -\frac{\cos mx}{m}, \quad \int \cos mx\,dx = \frac{\sin mx}{m}. \qquad (c)$$

$$\int \frac{dx}{\cos^2 x} = \tan x, \quad \int \frac{dx}{\sin^2 x} = -\cot x. \qquad (d)$$

---

[*] It was in this aspect that the process of integration was treated by Leibnitz, the symbol of integration ∫ being regarded as the initial letter of the word *sum*, in the same way as the symbol of differentiation d is the initial letter in the word *difference*.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}, \qquad (e)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}. \qquad (f)$$

$$\int e^x dx = e^x, \int a^x dx = \frac{a^x}{\log a}. \qquad (g)$$

These, together with two or three additional forms which shall be afterwards supplied, are called the *fundamental** or *elementary* integrals, to which all other forms,† that admit of integration in a finite number of terms, are ultimately reducible.

Many integrals are immediately reducible to one or other of these forms; a few simple examples are given for exercise.

EXAMPLES.

1. $\int \frac{dx}{\sqrt{x}}$. *Ans.* $2\sqrt{x}$.

2. $\int \tan x \, dx$. $-\log(\cos x)$.

3. $\int \frac{x^{n-1}dx}{a + bx^n}$. $\frac{1}{nb}\log(a + bx^n)$.

4. $\int \frac{x dx}{\sqrt{1 - x^2}}$. $-\sqrt{1 - x^2}$.

5. $\int \frac{dx}{a + bx^2}$. $\frac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\frac{b}{a}}\right)$.

6. $\int \frac{\sin\theta \, d\theta}{\cos^2\theta}$. $\sec\theta$.

---

* The fundamental integrals are denoted in this chapter by the letters *a*, *b*, *c*, &c.; the other formulæ by numerals 1, 2, 3, &c.

† By integrable forms are here understood those contained in the elementary portion of the Integral Calculus, as involving the ordinary transcendental functions only, and excluding what are styled Elliptic and Hyper-Elliptic functions.

7. $\int \frac{dx}{x^{\frac{4}{3}}}.$ *Ans.* $-\frac{3}{x^{\frac{1}{3}}}.$

8. $\int \frac{dx}{x^{\frac{n-1}{n}}}.$ $nx^{\frac{1}{n}}.$

9. $\int \frac{dx}{x-a}.$ $\log (x - a).$

3. **Integral of a Sum.**—It follows immediately from Art. 12, Dif. Cal., that the integral of the sum of any number of differentials is the sum of the integrals of each taken separately. For example—

$$\int (Ax^m + Bx^n + Cx^r + \&c.)\, dx = A\int x^m dx + B\int x^n dx + C\int x^r dx + \&c.$$

$$= \frac{Ax^{m+1}}{m+1} + \frac{Bx^{n+1}}{n+1} + \frac{Cx^{r+1}}{r+1} + \&c. \qquad (2)$$

Hence we can write down immediately the integral of any function which is reducible to a finite number of terms consisting of powers of x multiplied by constant coefficients.

Again, to find the integrals of $\cos^2 x dx$ and $\sin^2 x dx$; here

$$\int \cos^2 x dx = \int \frac{1 + \cos 2x}{2} dx = \frac{x}{2} + \frac{\sin 2x}{4}, \qquad (3)$$

$$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4}. \qquad (4)$$

A few examples are added for practice.

EXAMPLES.

1. $\int \frac{(1 - x^2)^2 dx}{x}.$ *Ans.* $\log x - x^2 + \frac{x^4}{4}.$

2. $\int \frac{(x - 2) dx}{x\sqrt{x}}.$ $2\sqrt{x} + \frac{4}{\sqrt{x}}.$

3. $\int \tan^2 x dx = \int (\sec^2 x - 1)\, dx$ $\tan x - x.$

4. $\int \cos mx \cos nx dx$. *Ans.* $\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}$.

5. $\int \sin mx \sin nx dx$. $\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$.

6. $\int \sqrt{\frac{a+x}{a-x}}\, dx$. $a \sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2}$.

Multiply the numerator and denominator by $\sqrt{a+x}$.

7. $\int x\sqrt{x+a}\, dx$. $\frac{2}{5}(x+a)^{\frac{5}{2}} - \frac{2}{3}a(x+a)^{\frac{3}{2}}$.

8. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$. $\frac{2}{3a}\left((x+a)^{\frac{3}{2}} - x^{\frac{3}{2}}\right)$.

Multiply the numerator and denominator by the complementary surd

$$\sqrt{x+a} - \sqrt{x}.$$

4. **Integration by Substitution.**—The integration of many expressions is immediately reducible to the elementary forms in Art. 2, by the substitution of a new variable.

For example, to integrate $(a + bx)^n dx$, we substitute z for $a + bx$; then $dz = bdx$, and

$$\int (a+bx)^n dx = \int \frac{z^n dz}{b} = \frac{z^{n+1}}{(n+1)b} = \frac{(a+bx)^{n+1}}{(n+1)b}.$$

Again, to find

$$\int \frac{x^2 dx}{(a+bx)^n},$$

we substitute z for $a + bx$, as before, when the integral becomes

$$\frac{1}{b^3}\int \frac{(z-a)^2\, dz}{z^n},$$

or $$-\frac{1}{b^3}\left\{\frac{1}{(n-3)z^{n-3}} - \frac{2a}{(n-2)z^{n-2}} + \frac{a^2}{(n-1)z^{n-1}}\right\}.$$

On replacing z by $a + bx$, the required integral can be expressed in terms of x.

The more general integral

$$\int \frac{x^m dx}{(a + bx)^n},$$

where m is any positive integer, by a like substitution becomes

$$\frac{1}{b^{n+1}} \int \frac{(z - a)^m \, dz}{z^n};$$

expanding by the binomial theorem and integrating each te separately, the required integral can be immediately o tained.

Again, to find

$$\int \frac{dx}{x^m (a + bx)^n},$$

we substitute z for $\frac{a}{x} + b$, and it becomes

$$-\frac{1}{a^{m+n-1}} \int \frac{(z - b)^{m+n-2} dz}{z^n},$$

which is integrable, as before, whenever $m + n$ is a positive integer greater than unity.

It may be observed that all fractional expressions in which the numerator is the differential of the denominator can be immediately integrated.

For we obviously have, from (b),

$$\int \frac{f'(x) \, dx}{f(x)} = \log f(x). \qquad (5)$$

EXAMPLES.

1. $\int \frac{\sin x \, dx}{a + b \cos x}$. *Ans.* $-\frac{\log(a + b \cos x)}{b}$.

2. $\int \frac{x^3 dx}{\sqrt{a^8 - x^8}}$. $\frac{1}{4} \sin^{-1} \left(\frac{x}{a}\right)^4$.

3. $\int \log x \frac{dx}{x}$. $\frac{1}{2} (\log x)^2$.

4. $\int \frac{dx}{x \log x}$. $\log (\log x)$.

5. $\int \frac{x^2 \, dx}{(a + bx)^3}$. $\frac{\log (a + bx)}{b^3} + \frac{3a^2 + 4abx}{2b^3 (a + bx)^2}$.

6. $\int \frac{dx}{x^2 (a + bx)^2}$. $\frac{2b}{a^3} \log \frac{a + bx}{x} - \frac{a + 2bx}{a^2 x (a + bx)}$.

7. $\int \frac{xdx}{(a + bx)^{\frac{1}{2}}}$. $\frac{2 (a + bx)^{\frac{3}{2}}}{3b^2} - \frac{2a(a + bx)^{\frac{1}{2}}}{b^2}$,

8. $\int \frac{xdx}{(a + bx)^{\frac{1}{3}}}$. $\frac{3 (a + bx)^{\frac{5}{3}}}{5b^2} - \frac{3a (a + bx)^{\frac{2}{3}}}{2b^2}$.

9. $\int \frac{dx}{x\sqrt{2ax - a^2}}$. $\frac{2}{a} \tan^{-1} \sqrt{\frac{2x - a}{a}}$.

Assume $2ax - a^2 = z^2$, then $adx = zdz$, and the transformed integral is

$$\int \frac{2dz}{a^2 + z^2}.$$

5. **Integration of** $\frac{dx}{x^2 - a^2}$.

Since
$$\frac{1}{x^2 - a^2} = \frac{1}{2a} \left\{\frac{1}{x - a} - \frac{1}{x + a}\right\},$$
we get
$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a}. \qquad (h)$$

This is to be regarded as another fundamental formula additional to those contained in Art. 2.

In like manner we have

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b} \log \frac{x-a}{x-b}. \qquad (6)$$

6. **Integration of**

$$\frac{dx}{a + 2bx + cx^2}.$$

This may be written in the form

$$\frac{c\,dx}{(cx+b)^2 + ac - b^2};$$

or, substituting z for $cx + b$,

$$\frac{dz}{z^2 + ac - b^2}.$$

This is of the form (f) or (h) according as $ac - b^2$ is positive or negative.

Hence, if $ac > b^2$ we have

$$\int \frac{dx}{a + 2bx + cx^2} = \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{cx + b}{\sqrt{ac - b^2}}. \qquad (7)$$

If $ac < b^2$,

$$\int \frac{dx}{x^2 + 2bx + cx^2} = \frac{1}{2\sqrt{b^2 - ac}} \log \frac{cx + b - \sqrt{b^2 - ac}}{cx + b + \sqrt{b^2 - ac}}. \qquad (8)$$

This latter form can also be immediately obtained from (6).

7. **Integration of**

$$\frac{(p + qx)\,dx}{a + 2bx + cx^2}.$$

This can at once be written in the form

$$\frac{q}{c} \cdot \frac{(b + cx)\,dx}{a + 2bx + cx^2} + \frac{pc - qb}{c} \cdot \frac{dx}{a + 2bx + cx^2}.$$

The integral of the first term is evidently

$$\frac{q}{2c}\log\left(a + 2bx + cx^2\right),$$

while the integral of the second is obtained by the preceding Article.

For example, let it be proposed to integrate

$$\frac{(x\cos\theta - 1)\,dx}{x^2 - 2x\cos\theta + 1}.$$

The expression becomes in this case

$$\frac{\cos\theta\,(x - \cos\theta)\,dx}{x^2 - 2x\cos\theta + 1} - \frac{\sin^2\theta dx}{(x - \cos\theta)^2 + \sin^2\theta},$$

hence

$$\int\frac{(x\cos\theta - 1)\,dx}{x^2 - 2x\cos\theta + 1} = \frac{\cos\theta}{2}\log\left(x^2 - 2x\cos\theta + 1\right)$$

$$- \sin\theta\tan^{-1}\frac{x - \cos\theta}{\sin\theta}. \qquad (9)$$

When the roots of $a + 2b + cx^2$ are real, it will be found simpler to integrate the expression by its decomposition into partial fractions. A general discussion of this method will be given in the next chapter.

EXAMPLES.

1. $\int\frac{dx}{1 + x + x^2}$. *Ans.* $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right)$.

2. $\int\frac{dx}{1 + x - x^2}$. $\frac{1}{\sqrt{5}}\log\left(\frac{2x - 1 + \sqrt{5}}{2x - 1 - \sqrt{5}}\right)$.

3. $\int\frac{dx}{x^2 + x - 12}$. $\frac{1}{7}\log\left(\frac{x - 3}{x + 4}\right)$.

4. $\int \frac{dx}{x^2 + 4x + 5}$. $\qquad \tan^{-1}(x + 2)$.

5. $\int \frac{dx}{5x^2 + 4x + 8}$. $\qquad \frac{1}{6} \tan^{-1} \frac{5x + 2}{6}$.

6. $\int \frac{x^2 dx}{1 - x^6}$. $\qquad \frac{1}{3} \log \left(\frac{1 + x^3}{1 - x^3} \right)$.

7. $\int \frac{x^3 dx}{x^8 - x^4 - 6}$. $\qquad \frac{1}{20} \log \left(\frac{x^4 - 3}{x^4 + 2} \right)$.

8. $\int \frac{dx}{1 - 2x + 2x^2}$. $\qquad \tan^{-1}(2x - 1)$.

8. **Exponential Values for** sin θ **and** cos θ.—By comparing the fundamental formulæ (f) and (h), the well-known exponential forms for sin θ and cos θ can be immediately deduced, as follows.

Substitute $z\sqrt{-1}$ for x in both sides of the equation

$$\int \frac{dx}{1 - x^2} = \frac{1}{2} \log \left(\frac{1 + x}{1 - x} \right) + const.;$$

and we get

$$\int \frac{dz}{1 + z^2} = \frac{1}{2\sqrt{-1}} \log \left(\frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}} \right) + const.$$

or, by (f), $\qquad \tan^{-1} z = \frac{1}{2\sqrt{-1}} \log \left(\frac{1 + z\sqrt{-1}}{1 - z\sqrt{-1}} \right) + const.$

Now, let $z = \tan\theta$ and this becomes

$$\theta = \frac{1}{2\sqrt{-1}} \log \left(\frac{1 + \sqrt{-1} \tan\theta}{1 - \sqrt{-1} \tan\theta} \right) + const.$$

When $\theta = 0$, this reduces to $0 = const.$

Hence $$e^{2\theta\sqrt{-1}} = \frac{\cos\theta + \sqrt{-1}\sin\theta}{\cos\theta - \sqrt{-1}\sin\theta} = (\cos\theta + \sqrt{-1}\sin\theta)^2,$$

or, $$e^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta,$$

$$e^{-\theta\sqrt{-1}} = \cos\theta - \sqrt{-1}\sin\theta,$$

9. **Integration of** $\dfrac{dx}{\sqrt{x^2 \pm a^2}}$.

Assume* $$\sqrt{x^2 \pm a^2} = z - x.$$

then we get $$\pm a^2 = z^2 - 2xz,$$

hence $$(z - x)\,dz = z\,dx, \text{ or } \frac{dx}{z - x} = \frac{dz}{z};$$

$$\therefore \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \int \frac{dz}{z} = \log z = \log(x + \sqrt{x^2 \pm a^2}). \qquad (i)$$

This is to be regarded as another fundamental form.

By aid of this and form (*e*) it is evident that all expressions of the shape

$$\frac{dx}{\sqrt{a + 2bx + cx^2}}$$

can be immediately integrated; a, b, c, being any constants, positive or negative.

The preceding integration evidently depends on formula (*i*), or (*e*), according as the coefficient of x^2 is positive or negative.

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* The student will better understand the propriety of this assumption after reading a subsequent chapter, in which a general transformation, of which the above is a particular case, will be given.

Thus, we have

$$\int \frac{dx}{\sqrt{a+2bx+cx^2}} = \frac{1}{\sqrt{c}} \log\left(x\sqrt{c} + \frac{b}{\sqrt{c}}\ \sqrt{a+2bx+cx^2}\right). \quad (10)$$

$$\int \frac{dx}{\sqrt{a+2bx-cx^2}} = \frac{1}{\sqrt{c}} \sin^{-1}\left(\frac{cx-b}{\sqrt{ac+b^2}}\right). \quad (11)$$

c being regarded as positive in both integrals.

When the factors in the quadratic $a + 2bx + cx^2$ are real and given, the preceding integral can be exhibited in a simpler form by the method of the two next Articles.

10. **Integration of** $\dfrac{dx}{\sqrt{(x+a)(x+\beta)}}$.

Assume $\qquad x + a = z^2$, then $dx = 2zdz$,

$$\therefore \frac{dx}{\sqrt{x+a}} = 2dz,$$

hence

$$\frac{dx}{\sqrt{(x+a)(x+\beta)}} = \frac{2dz}{\sqrt{z^2-a+\beta}};$$

$$\therefore \int \frac{dx}{\sqrt{(x+a)(x+\beta)}} = 2\int \frac{dz}{\sqrt{z^2-a+\beta}}$$

$$= 2 \log\left(z + \sqrt{z^2-a+\beta}\right), \text{ by } (i),$$

or

$$\int \frac{dx}{\sqrt{(x+a)(x+\beta)}} = 2 \log\left(\sqrt{x+a} + \sqrt{x+\beta}\right). \quad (12)$$

11. **Integration of** $\dfrac{dx}{\sqrt{(x+a)(\beta-x)}}$.

As before, assume $x + a = z^2$, and we get

$$\frac{dx}{\sqrt{(x+a)(\beta-x)}} = \frac{2dz}{\sqrt{\beta+a-z^2}}.$$

Hence, by (*e*),

$$\int \frac{dx}{\sqrt{(x+a)(\beta-x)}} = 2\sin^{-1}\sqrt{\frac{x+a}{\beta+a}}. \qquad (13)$$

12. Again, as in Art. 7, the expression

$$\frac{(p+qx)\,dx}{\sqrt{a+2bx+cx^2}}$$

can be transformed into

$$\frac{q}{c}\,\frac{(b+cx)\,dx}{\sqrt{a+2bx+cx^2}} + \frac{pc-qb}{c}\,\frac{dx}{\sqrt{a+2bx+cx^2}},$$

and is, accordingly, immediately integrable by aid of the preceding formulæ.

EXAMPLES.

1. $\int \frac{dx}{\sqrt{x^2-ax}}$. *Ans.* $2\log(\sqrt{a}+\sqrt{x-a})$.

2. $\int \frac{dx}{\sqrt{ax-x^2}}$. $2\sin^{-1}\sqrt{\frac{x}{a}}$.

3. $\int \frac{dx}{\sqrt{3x-x^2-2}}$. $2\sin^{-1}\sqrt{\frac{x-1}{3}}$.

4. $\int \frac{dx}{\sqrt{1+x+x^2}}$. $\log(2x+1+2\sqrt{1+x+x^2})$.

5. $\int \sqrt{\frac{x+a}{x+b}}\,dx = \sqrt{(x+a)(x+b)} + (a-b)\log(\sqrt{x+a}+\sqrt{x+b})$.

Multiply the numerator and denominator by $\sqrt{x+a}$.

6. $\int \frac{dx}{\sqrt{1-x-x^2}}$. *Ans.* $\sin^{-1}\frac{2x+1}{\sqrt{5}}$.

7. $\int \frac{dx}{\sqrt{(a+bx)(a'-b'x)}}$. $\frac{2}{\sqrt{bb'}}\sin^{-1}\sqrt{\frac{b'(a+bx)}{ab'+ba'}}$.

13. **Integration of** $\dfrac{dx}{(x+p)\sqrt{a+2bx+cx^2}}$.

Let $x+p=\dfrac{1}{z}$, then

$$\frac{dx}{x+p}=-\frac{dz}{z},\text{ and }x=\frac{1-pz}{z}.$$

$$\therefore \int\frac{dx}{(x+p)\sqrt{a+2bx+cx^2}}=\int\frac{-dz}{\sqrt{az^2+2bz(1-pz)+c(1-pz)^2}}$$

$$=\int\frac{-dz}{\sqrt{a'+2b'z+c'z^2}};$$

where $\qquad a'=c,\ b'=b-cp,\ c'=a-2bp+cp^2.$

The integral consequently is reducible to (10), or (11), according as c' is positive or negative.

EXAMPLES.

1.	$\displaystyle\int\frac{dx}{x\sqrt{x^2-a^2}}.$	*Ans.* $\dfrac{1}{a}\cos^{-1}\left(\dfrac{a}{x}\right).$
2.	$\displaystyle\int\frac{dx}{x\sqrt{x^2+1}}.$	$\log\left(\dfrac{\sqrt{1+x^2-x}}{x}\right).$
3.	$\displaystyle\int\frac{dx}{(1+x)\sqrt{1-x^2}}.$	$-\sqrt{\dfrac{1-x}{1+x}}.$
4.	$\displaystyle\int\frac{dx}{x\sqrt{a+2bx+cx^2}}.$	$\dfrac{1}{\sqrt{a}}\log\left(\dfrac{x}{a+bx+\sqrt{a}\sqrt{a+2bx+cx^2}}\right).$
5.	$\displaystyle\int\frac{dx}{x\sqrt{cx^2+2bx-a}}.$	$\dfrac{1}{\sqrt{a}}\sin^{-1}\left(\dfrac{bx-a}{x\sqrt{ac+b^2}}\right).$
6.	$\displaystyle\int\frac{dx}{(1+x)\sqrt{1+2x-x^2}}.$	$\dfrac{1}{\sqrt{2}}\sin^{-1}\left(\dfrac{x\sqrt{2}}{1+x}\right).$
7.	$\displaystyle\int\frac{dx}{(1+x)\sqrt{1+2x-x^2}}.$	$\sin^{-1}\left(\dfrac{1+3x}{(1+x)\sqrt{5}}\right).$

14. The transformation adopted in the last Article is one of frequent application in Integration. It is, accordingly, worthy of the student's notice that when we change x into $\frac{1}{z}$ we have $\frac{dx}{x} = -\frac{dz}{z}$; and, in general, if $x^n = \frac{1}{z}$, $\frac{dx}{x} = -\frac{dz}{nz}$.

These results follow immediately from *logarithmic* differentiation, and often furnish a clue as to when an Integration is facilitated by such a transformation.

For example, let us take the integral—

$$\int \frac{dx}{x\,(a+bx^n)}.$$

Here, the substitution of $\frac{1}{z}$ for x^n gives

$$-\frac{1}{n}\int \frac{dz}{az+b}.$$

The value of which is obviously

$$-\frac{1}{na}\log\,(az+b),\ \text{or}\ \frac{1}{na}\log\left(\frac{x^n}{a+bx^n}\right).$$

15. **Integration of** $\frac{dx}{(a+cx^2)^{\frac{3}{2}}}$.

Let $x = \frac{1}{z}$, and the expression becomes

$$-\frac{zdz}{(az^2+c)^{\frac{3}{2}}};$$

the integral of this is evidently

$$\frac{1}{a\,(az^2+c)^{\frac{1}{2}}},\ \text{or}\ \frac{x}{a\,(a+cx^2)^{\frac{1}{2}}}.$$

Hence $$\frac{dx}{(a+cx^2)^{\frac{3}{2}}} = \frac{x}{a\,(a+cx^2)^{\frac{1}{2}}}. \qquad (14)$$

16. More generally, to find the integral of

$$\frac{dx}{(a + 2bx + cx^2)^{\frac{3}{2}}}.$$

This can be written in the form

$$\frac{c^{\frac{3}{2}}dx}{(ac - b^2 + (cx + b)^2)^{\frac{3}{2}}},$$

which is reduced to the preceding on making $cx + b = z$.

Hence, we get

$$\int \frac{dx}{(a + 2bx + cx^2)^{\frac{3}{2}}} = \frac{b + cx}{(ac - b^2)\ (a + 2bx + cx)^{\frac{1}{2}}}. \quad (15)$$

Again, if we substitute $\frac{1}{z}$ for x,

$$\frac{xdx}{(a + 2bx + cx^2)^{\frac{3}{2}}} \text{ becomes } \frac{-dz}{(az^2 + 2bz + c)^{\frac{3}{2}}},$$

and, accordingly, we have

$$\int \frac{xdx}{(a + 2bx + cx^2)^{\frac{3}{2}}} = -\frac{a + bx}{(ac - b^2)\ (a + 2bx + cx^2)^{\frac{1}{2}}}. \quad (16)$$

Combining these two results, we get

$$\int \frac{(p + qx)\,dx}{(a + 2bx + cx^2)^{\frac{3}{2}}} = \frac{bp - aq + (cp - bq)\,x}{(a + 2bx + cx^2)^{\frac{1}{2}}}.$$

17. **Integration of** $\frac{d\theta}{\sin\theta}$ **and** $\frac{d\theta}{\cos\theta}$.

It will be shown in a subsequent chapter that the integration of a numerous class of expressions is reducible either to that of $\frac{d\theta}{\sin\theta}$, or of $\frac{d\theta}{\cos\theta}$; we accordingly propose to investigate their values here. For this purpose we shall first find the integral of $\frac{d\theta}{\sin\theta\cos\theta}$.

Here $$\frac{d\theta}{\sin\theta\cos\theta} = \frac{\dfrac{d\theta}{\cos^2\theta}}{\tan\theta} = \frac{d(\tan\theta)}{\tan\theta}.$$

Consequently $$\int\frac{d\theta}{\sin\theta\cos\theta} = \log(\tan\theta). \qquad (17)$$

Next, to find the integral of

$$\frac{d\theta}{\sin\theta}.$$

This can be written in the form

$$\frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}},$$

and by the preceding we have

$$\int\frac{d\theta}{\sin\theta} = \log\left(\tan\frac{\theta}{2}\right). \qquad (18)$$

Again, to determine the integral of $\frac{d\theta}{\cos\theta}$, we substitute $\frac{\pi}{2} - \phi$ for θ, and the expression becomes $\frac{-d\phi}{\sin\phi}$; the integral of this, by (18), is

$$-\log\left(\tan\frac{\phi}{2}\right), \text{ or } \log\left(\cot\frac{\phi}{2}\right), \text{ or } \log\left\{\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right\}.$$

Accordingly, we have

$$\int\frac{d\theta}{\cos\theta} = \log\left\{\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right\} = \log\left\{\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}. \qquad (19)$$

This integral can also be easily obtained otherwise, as follows :—

$$\int\frac{d\theta}{\cos\theta} = \int\frac{\cos\theta\, d\theta}{\cos^2\theta} = \int\frac{d(\sin\theta)}{\cos^2\theta}.$$

Let $\sin\theta = x$, and the integral becomes

$$\int \frac{dx}{1 - x^2} = \frac{1}{2}\log\left(\frac{1 + x}{1 - x}\right) = \frac{1}{2}\log\left(\frac{1 + \sin\theta}{1 - \sin\theta}\right).$$

The student will find no difficulty in identifying this result with that contained in (19).

18. **Integration of** $\dfrac{d\theta}{a + b\cos\theta}$.

This can be immediately written in the form

$$\frac{d\theta}{(a + b)\cos^2\frac{\theta}{2} + (a - b)\sin^2\frac{\theta}{2}},$$

or
$$\frac{\sec^2\frac{\theta}{2}\,d\theta}{a + b + (a - b)\tan^2\frac{\theta}{2}};$$

on substituting z for $\tan\frac{\theta}{2}$ this becomes

$$\frac{2dz}{a + b + (a - b)z^2}.$$

Consequently, by Ex. 5, Art. 2, we get

(1). when $a > b$,

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left\{\sqrt{\frac{a - b}{a + b}}\tan\frac{\theta}{2}\right\}. \qquad (20)$$

(2), when $a < b$, by formula (h),

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{2}{\sqrt{b^2 - a^2}} \log \left\{ \frac{\sqrt{b+a} + \sqrt{b-a}\tan\frac{\theta}{2}}{\sqrt{b+a} - \sqrt{b-a}\tan\frac{\theta}{2}} \right\}. \quad (21)$$

If we assume $a = b\cos\alpha$, the latter integral is easily transformed into

$$\frac{1}{b}\int \frac{d\theta}{\cos\alpha + \cos\theta} = \frac{1}{b\sin\alpha} \log \left\{ \frac{\cos\frac{\alpha - \theta}{2}}{\cos\frac{\alpha + \theta}{2}} \right\}.$$

The integral in (20) can be transformed into

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \left\{ \frac{b + a\cos\theta}{a + b\cos\theta} \right\}.$$

In a subsequent chapter a more general class of integrals which depend on the preceding will be discussed.

19. **Methods of Integration.**—The reduction of the integration of functions to one or other of the fundamental formulæ is usually effected by one of the following methods :—

(1). Transformation by the introduction of a new variable.
(2). Integration by parts.
(3). Integration by rationalization.
(4). Successive reduction.
(5). Decomposition into partial fractions.

Two or more of these methods can often be combined with advantage. It may also be observed that these different methods are not essentially distinct; thus the method of rationalization is a case of the first method, as it is always effected by the substitution of a new variable.

We proceed to illustrate these processes by a few elementary examples, reserving their fuller treatment for subsequent consideration.

20. **Integration by Transformation.**—Examples of this method have been already given in Arts. 4, 10, &c. One or two additional cases are here added.

Ex. 1. To find the integral of $\sin^2 x \cos^3 x \, dx$.

Let $\sin x = y$, and the transformed integral is

$$\int y^2 (1 - y^2)\, dy = \int y^2 dy - \int y^4 dy = \frac{y^3}{3} - \frac{y^5}{5} = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}.$$

Ex. 2. $$\int \frac{e^x dx}{1 + e^{2x}}.$$

Let $e^x = y$, and we get

$$\int \frac{dy}{1 + y^2} = \tan^{-1} y = \tan^{-1}(e^x).$$

21. **Integration by Parts.**—We have seen in Art. 13 Dif. Cal. that

$$d(uv) = udv + vdu;$$

hence we get

$$uv = \int udv + \int vdu,$$

or $$\int udv = uv - \int vdu. \qquad (22)$$

Consequently the integration of an expression of the form udv can be always made to depend on that of the expression vdu.

The advantage of this method will be best explained by applying it to a few elementary cases.

Ex. 1. $$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x dx}{\sqrt{1 - x^2}}.$$

$$= x \sin^{-1} x + \sqrt{1 - x^2}.$$

(2), when $a < b$, by formula (h),

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{2}{\sqrt{b^2 - a^2}} \log \left\{ \frac{\sqrt{b+a} + \sqrt{b-a}\tan\frac{\theta}{2}}{\sqrt{b+a} - \sqrt{b-a}\tan\frac{\theta}{2}} \right\}. \quad (21)$$

If we assume $a = b\cos\alpha$, the latter integral is easily transformed into

$$\frac{1}{b}\int \frac{d\theta}{\cos\alpha + \cos\theta} = \frac{1}{b\sin\alpha} \log \left\{ \frac{\cos\frac{\alpha - \theta}{2}}{\cos\frac{\alpha + \theta}{2}} \right\}.$$

The integral in (20) can be transformed into

$$\int \frac{d\theta}{a + b\cos\theta} = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1} \left\{ \frac{b + a\cos\theta}{a + b\cos\theta} \right\}.$$

In a subsequent chapter a more general class of integrals which depend on the preceding will be discussed.

19. **Methods of Integration.**—The reduction of the integration of functions to one or other of the fundamental formulæ is usually effected by one of the following methods:—

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Two or more of these methods can often be combined with advantage. It may also be observed that these different methods are not essentially distinct; thus the method of rationalization is a case of the first method, as it is always effected by the substitution of a new variable.

We proceed to illustrate these processes by a few elementary examples, reserving their fuller treatment for subsequent consideration.

EXAMPLES.

1. $\int x^n \log x dx.$ *Ans.* $\frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right).$

2. $\int \tan^{-1}x \, dx.$ $x \tan^{-1}x - \frac{1}{2}\log(1 + x^2).$

3. $\int x \tan^2 x \, dx.$ $x \tan x + \log(\cos x) - \frac{x^2}{2}.$

4. $\int \frac{\sin^{-1} x \, dx}{(1 - x^2)^{\frac{3}{2}}}.$ $\frac{x \sin^{-1} x}{\sqrt{1 - x^2}} + \frac{1}{2}\log(1 - x^2).$

Let $x = \sin y$, and the integral becomes

$$\int dy \frac{y}{\cos^2 y} = \int y \, d(\tan y) = y \tan y + \log(\cos y).$$

5. $\int e^x x^2 dx.$ $e^x(x^2 - 2x + 2).$

22. **Integration by Rationalization.**—By a proper assumption of a new variable we can, in many cases, change an irrational expression into a rational one, and thus integrate it. An instance of this method has been given in Art. 8.

The simplest case is where the quantity under the radical sign is of the form $a + bx$; such expressions admit of being easily integrated.

For example, let the expression be of the form

$$\frac{x^n dx}{(a + bx)^{\frac{1}{2}}},$$

where n is a positive integer. Suppose $a + bx = z^2$, then

$$dx = \frac{2z dz}{b} \text{ and } x = \frac{z^2 - a}{b};$$

making these substitutions, the expression becomes

$$\frac{2(z^2 - a)^n dz}{b^{n+1}}.$$

Ex. 2. $$\int x \log x \, dx.$$

Let $u = \log x$, $v = \frac{x^2}{2}$, and we get

$$\int x \log x \, dx = \frac{x^2 \log x}{2} - \frac{1}{2}\int x^2 \frac{dx}{x} = \frac{x^2}{2}\left(\log x - \frac{1}{2}\right).$$

Ex. 3. $$\int e^{ax} x dx.$$

Let $x = u$, $\frac{e^{ax}}{a} = v$, then

$$\int xe^{ax} \, dx = \frac{xe^{ax}}{a} - \int \frac{e^{ax}}{a} \, dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right).$$

Ex. 4. $$\int e^{ax} \sin mx \, dx.$$

Let $\sin mx = u$, $\frac{e^{ax}}{a} = v$, then

$$\int e^{ax} \sin mx \, dx = \frac{e^{ax} \sin mx}{a} - \frac{m}{a}\int e^{ax} \cos mx \, dx.$$

imilarly, $$\int e^{ax} \cos mx \, dx = \frac{e^{ax} \cos mx}{a} + \frac{m}{a}\int e^{ax} \sin mx \, dx.$$

ubstituting, and solving for $\int e^{ax} \sin mx \, dx$, we obtain

$$\int e^{ax} \sin mx \, dx = \frac{e^{ax}(a \sin mx - m \cos mx)}{a^2 + m^2}. \qquad (23)$$

In like manner we get

$$\int e^{ax} \cos mx \, dx = \frac{e^{ax}(a \cos mx + m \sin mx)}{a^2 + m^2}. \qquad (24)$$

Again, if we make $x = \frac{1}{z}$, the expression

$$\frac{dx}{x^n (a + cx^2)^{\frac{1}{2}}}$$

transforms into

$$\frac{z^{n-1}\, dz}{(az^2 + c)^{\frac{1}{2}}};$$

and is reducible to the preceding form when n is an *even positive* integer.

Hence, in this case, the expression can be easily integrated by the substitution $(a + cx^2)^{\frac{1}{2}} = xy$.

It will be subsequently seen that the integrals discussed in this and the preceding Articles are cases of a more general form, which is integrable by a similar transformation.

EXAMPLES.

1. $\int \frac{dx}{x^4 (x^2 - 1)^{\frac{1}{2}}}$. *Ans.* $\frac{\sqrt{x^2 - 1}}{3x^3} (2x^2 + 1)$.

2. $\int \frac{dx}{x^6 (1 + x^2)^{\frac{1}{2}}}$. $\frac{(x^2 + 1)^{\frac{1}{2}}}{5x} \left\{\frac{8}{3} + \frac{4}{x^2} + \frac{1}{x^4}\right\}$.

25. **Integration of** $\frac{dx}{(A + Cx^2)\, (a + cx^2)^{\frac{1}{2}}}$.

As in the preceding Art., let $(a + cx^2)^{\frac{1}{2}} = xz$, or $a + cx^2 = x^2z^2$: then, if we differentiate and divide by $2x$, we shall have

$$cdx = z^2dx + xz\, dz, \text{ or } \frac{dx}{xz} = \frac{dz}{c - z^2},$$

$$\therefore \frac{dx}{(a + cx^2)^{\frac{1}{2}}} = \frac{dz}{c - z^2}; \qquad (25)$$

and the transformed expression evidently is

$$\frac{dz}{(Ac - Ca) - Az^2}.$$

This is reducible to the fundamental formula (*f*), or (*h*), according as $\frac{Ac - Ca}{A}$ is positive or negative.

Hence, (1) if $\frac{Ac - Ca}{A} > 0$, the integral is easily seen to be

$$\frac{1}{2\sqrt{A(Ac - Ca)}} \log \left(\frac{\sqrt{A(a + cx^2)} + x\sqrt{Ac - Ca}}{\sqrt{A(a + cx^2)} - x\sqrt{Ac - Ca}} \right). \qquad (26)$$

(2). If $\frac{Ac - Ca}{A} < 0$, the value of the integral is

$$\frac{1}{\sqrt{A(Ca - Ac)}} \tan^{-1} \frac{x\sqrt{Ca - Ac}}{\sqrt{A(a + cx^2)}}. \qquad (27)$$

EXAMPLES.

1. $\int \frac{dx}{(1 + x^2)(1 - x^2)^{\frac{1}{2}}}$ *Ans.* $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x\sqrt{2}}{\sqrt{1 - x^2}} \right)$

2. $\int \frac{dx}{(3 + 4x^2)(4 - 3x^2)^{\frac{1}{2}}}$. $\frac{1}{5\sqrt{3}} \tan^{-1} \left(\frac{5x}{\sqrt{12 - 9x^2}} \right)$.

3. $\int \frac{dx}{(4 - 3x^2)(3 + 4x^2)^{\frac{1}{2}}}$. $\frac{1}{20} \log \frac{2\sqrt{3 + 4x^2} + 5x}{2\sqrt{3 + 4x^2} - 5x}$.

26. **Rationalization by Trigonometrical Transformation.**—It can be easily seen, as in Art. 6, that the irrational expression $\sqrt{a + 2bx + cx^2}$ can be always transformed into one or other of the following shapes.

$$(1)\ (a^2 - z^2)^{\frac{1}{2}}:\ (2),\ (a^2 + z^2)^{\frac{1}{2}};\ (z^2 - a^2)^{\frac{1}{2}};$$

neglecting a constant multiplier in each case.

Accordingly, any algebraic expression in x which contains one, and but one, surd of a quadratic form, is capable of being rationalized by a trigonometrical transformation. The first of the forms, by making $z = a \sin \theta$; the second, by $z = a \tan \theta$; and the third, by $z = a \sec \theta$.

For, (1). When $z = a \sin\theta$, we have $(a^2 - z^2)^{\frac{1}{2}} = a\cos\theta$, and $dz = a\cos\theta d\theta$.

(2). When $z = a\tan\theta$, ... $(a^2 + z^2)^{\frac{1}{2}} = a\sec\theta$, and $dz = \dfrac{a d\theta}{\cos^2\theta}$.

(3). When $z = a\sec\theta$, ... $(z^2 - a^2)^{\frac{1}{2}} = a\tan\theta$, and $dz = a\tan\theta\sec\theta d\theta$.

A number of integrations can be performed by aid of one or other of these transformations. In a subsequent place this class of transformations will be again considered. For the present we shall merely illustrate the method by a few examples.

EXAMPLES.

1. $$\int \frac{dx}{x^2(1 + x^2)^{\frac{1}{2}}}.$$

Let $x = \tan\theta$, and the integral becomes

$$\int \frac{\cos\theta\, d\theta}{\sin^2\theta} = \int \frac{d(\sin\theta)}{\sin^2\theta} = -\frac{1}{\sin\theta} = -\frac{\sqrt{1+x^2}}{x}.$$

2. $$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Let $x = a\sin\theta$, and we get

$$\frac{1}{a^2}\int \frac{d\theta}{\cos^2\theta} = \frac{\tan\theta}{a^2} = \frac{x}{a^2\sqrt{a^2 - x^2}}.$$

This has been integrated by another transformation in Art. 15.

3. $$\int \frac{dx}{x^3(x^2 - 1)^{\frac{1}{2}}}.$$

Let $x = \sec\theta$, and the integral becomes

$$\int \cos^2\theta\, d\theta, \text{ or, by (3), Art. 3, } \frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2}:$$

accordingly, the value of the integral in question is

$$\frac{\sqrt{x^2 - 1}}{2x^2} + \frac{1}{2}\cos^{-1}\left(\frac{1}{x}\right).$$

4. $$\int \frac{e^{a\tan^{-1}x}\,dx}{(1+x^2)^{\frac{3}{2}}}.$$

Let $x = \tan\theta$, and we get

$$\int \cos\theta\, e^{a\theta}\, d\theta\,;\ \text{or, by (23)},\ \frac{e^{a\theta}(a\cos\theta + \sin\theta)}{1+a^2}.$$

Hence, $$\int \frac{dx\, e^{a\tan^{-1}x}}{(1+x^2)^{\frac{3}{2}}} = \frac{(a+x)\, e^{a\tan^{-1}x}}{(1+a^2)(1+x^2)^{\frac{1}{2}}}.$$

5. $$\int dx \sin^{-1}\left(\frac{x}{a+x}\right)^{\frac{1}{2}}.$$

Let $\dfrac{x}{a+x} = \sin^2\theta$, or $x = a\tan^2\theta$, and the integral becomes

$$a\int \theta\, d(\tan^2\theta),\ \text{or}\ a\int \theta\, d(\sec^2\theta):\ (\text{since}\ \sec^2\theta = 1 + \tan^2\theta).$$

Integrating by parts, we have

$$\int \theta\, d(\sec^2\theta) = \theta\sec^2\theta - \int \sec^2\theta\, d\theta = \theta\sec^2\theta - \tan\theta:$$

hence the value of the proposed integral is

$$(a+x)\tan^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}} - (ax)^{\frac{1}{2}}.$$

It may be observed that the fundamental formulæ (*e*) and (*f*), can be at once obtained by aid of the transformations of this Article.

27. The student must not, however, take for granted that whenever one or other of the preceding transformations is applicable, it furnishes the simplest method of integration. We have, in Arts. 8 and 13, already met with such integrals, and have treated them by other substitutions; all that can be stated is, that the method given in the preceding Article will be often found the most simple and useful. The most suitable transformation in each case can only be arrived at after considerable practice and familiarity with the results introduced by such transformations.

By employing different methods we often obtain integrals of the same expression, which appear at first sight not to agree. On examination, however, it will always be found that they only differ by some constant; otherwise, they could not have the same differential.

28. **Higher Transcendental Functions.**—Whenever the expression under the radical sign contains powers of x beyond the second, the integral cannot, unless in exceptional cases, be reduced to any of the fundamental formulæ; and consequently cannot be represented in finite terms of x, or of the ordinary transcendental functions; i. e. logarithmic, exponential, or circular functions. Accordingly, the investigation of such integrals necessitates the introduction of higher classes of transcendental functions.

Thus the integration of irrational functions of x, in which the expression under the square root is of the third or fourth degree in x, depends on a higher class of transcendentals called Elliptic Functions.

29. The method of integration by successive reduction is reserved for a subsequent place. The integration of rational fractions by the method of decomposition into partial fractions will be considered in the next Chapter.

30. **Observations on Fundamental Forms.**—From what has been already stated, the sign of integration $\int$ may be regarded in the light of a question: viz. the meaning of the expression $\int F(x)\,dx$ is the same as asking what function of x has $F(x)$ for its first derived. The answer to this question can only be derived from our previous knowledge of the differential coefficients of the different classes of functions, as obtained by aid of the Differential Calculus. The number of fundamental formulæ of integration must accordingly, ultimately, be the same as the number of independent kinds of functions in Algebra and Trigonometry. These may be briefly classed as follows:—

(1). Ordinary powers and roots, such as x^m, $x^{\frac{p}{q}}$, &c.
(2). Exponentials, a^x, &c., and their inverse functions; viz. Logarithms.
(3). Trigonometric functions, sin x, tan x, &c., and their inverse functions; $\sin^{-1}x$, $\tan^{-1}x$, &c.

This classification may assist the student towards understanding why an expression, in order to be capable of integration in a finite form, in terms of x and the ordinary transcendental functions, must be reducible by transformation to one or other of the fundamental formulæ given in this chapter. He will also soon find that the classes of integrals which are so reducible are very limited, and that the large majority of expressions can only be integrated by the aid of infinite series.

The student must not expect to understand at once the *reason* for each transformation which he finds given: as he however gains familiarity with the subject he will find that most of the elementary integrations which can be performed group themselves under a few heads; and that the proper transformations are in general simple, not numerous, and usually not difficult to arrive at. He must often be prepared to abandon the transformation which seemed at first sight the most suitable: such failures are not, however, to be considered as waste of time, for it is by the application of such processes only that the student is enabled gradually to arrive at the general principles according to which integrals may be classified.

Many expressions will be found to admit of integration in two or more different ways. Such modes of arriving at the same results mutually throw light on each other, and will be found an instructive exercise for the beginner.

EXAMPLES.

1. $\int \frac{(1+\cos x)\,dx}{(x+\sin x)^3}$. *Ans.* $-\frac{1}{2}\frac{1}{(x+\sin x)^2}$

2. $\int x\sin x\,dx$. $\sin x - x\cos x$.

3. $\int \frac{1-x}{1+x}\,dx$. $2\log(1+x) - x$.

4. $\int (a+bx^n)^m x^{n-1}\,dx$. $\frac{(a+bx^n)^{m+1}}{n(m+1)b}$.

5. $\int \frac{x^2dx}{(a^3+x^3)^{\frac{3}{2}}}$. $\frac{2}{3}\frac{1}{(a^3+x^3)^{\frac{1}{2}}}$

6. $\int \frac{dx}{(1+x^2)\tan^{-1}x}$. $\log(\tan^{-1}x)$.

7. $\int \frac{dx}{\sqrt{5+4x-x^2}}$. $2\sin^{-1}\sqrt{\frac{x+1}{6}}$.

8. $\int \frac{x^2\,dx}{x^6+x^3-2}$. $\frac{1}{9}\log\left(\frac{x^3-1}{x^3+2}\right)$.

9. $\int \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$. $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right)$.

10. $\int \frac{\tan x\,dx}{a+b\tan^2 x}$. $\frac{1}{2(b-a)}\log a\,(a\cos^2 x+b\sin^2 x)$.

11. $\int \frac{\cos(\log x)\,dx}{x}$. $\sin(\log x)$.

12. Show that the integral of $\frac{dx}{x}$ can be obtained from that of $x^m\,dx$.

Write the integral of $x^m\,dx$ in the form $\frac{x^{m+1}-a^{m+1}}{m+1}$; and by the method of indeterminate forms, Ex. 5, Ch. iv. Dif. Cal., it can easily be seen that the true value of the fraction, when $m+1=0$, is $\log\left(\frac{x}{a}\right)$, or $\log x$, omitting the arbitrary constant.

13. $\int e^{ax}\sin mx\cos nx\,dx$.

This is immediately reducible to the integral given in formula (23).

14. $\int \frac{dx}{5+4\sin x}$. *Ans.* $\frac{2}{3}\tan^{-1}\left(\frac{4+5\tan\frac{x}{2}}{3}\right)$.

15. $\displaystyle\int \frac{x\, e^{a \tan^{-1} x}\, dx}{(1 + x^2)^{\frac{3}{2}}}.$ *Ans.* $\displaystyle\frac{e^{a \tan^{-1} x}(ax + 1)}{(1 + a^2)(1 + x^2)^{\frac{1}{2}}}.$

16. $\displaystyle\int x\,(a + x)^{\frac{1}{3}}\, dx.$ $\displaystyle\frac{3(a + x)^{\frac{4}{3}}(4x - 8a)}{4 \cdot 7}$

17. $\displaystyle\int \frac{x^3\, dx}{(a + bx^2)^{\frac{3}{2}}}.$ $\displaystyle\frac{2a + bx^2}{b^2 (a + bx^2)^{\frac{1}{2}}}.$

Let $a + bx^2 = z^2$.

18. $$\int \frac{(p + q \cos x)\, dx}{a + b \cos x}.$$

This is equivalent to

$$\int \frac{q\, dx}{b} + \frac{pb - qa}{b} \int \frac{dx}{a + b \cos x}.$$

and accordingly, can be integrated by Art. 18.

19. $\displaystyle\int \frac{x\, e^x\, dx}{(1 + x)^2}.$ *Ans.* $\displaystyle\frac{e^x}{1 + x}.$

20. $\displaystyle\int \frac{x\, dx}{1 + x^4}.$ $\displaystyle\frac{1}{2}\tan^{-1}(x^2).$

21. $\displaystyle\int \frac{x^2\, dx}{(a + bx)^{\frac{5}{2}}}.$ $\displaystyle\frac{x^3}{3a\,(a + bx^2)^{\frac{3}{2}}}.$

22. $\displaystyle\int \frac{dx}{x\sqrt{x^3 + 1}}.$ $\displaystyle\frac{2}{3}\log\left(\frac{\sqrt{1 + x^3} - 1}{\sqrt{1 + x^3} - 1}\right).$

Let $x^3 + 1 = z^2$.

23. $\displaystyle\int \frac{dx}{x\sqrt{x^n + 1}}.$ $\displaystyle\frac{2}{n}\log\frac{\sqrt{1 + x^n} - 1}{\sqrt{1 + x^n} - 1}.$

24. Integrate $$\frac{d\theta}{a + b \cos \theta}.$$

by aid of the assumption $$x = \frac{b + a \cos \theta}{a + \cos \theta}$$

25. Deduce Gregory's expansion for $\tan^{-1} x$ from formula (f). When $x < 1$, we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \&c.$$

$$\therefore \quad \tan^{-1}x = \int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c.$$

No constant is added since $\tan^{-1} x$ vanishes along with x.

26. Deduce in a similar manner the expansions of $\log(1+x)$ and $\sin^{-1} x$.

27. Find the integral of $\dfrac{d\theta}{a + b\cos\theta + c\sin\theta}$.

This can be reduced to the form in Art. 18, by assuming $\dfrac{b}{c} = \cot \alpha$, &c.

28. $\displaystyle\int \frac{dx}{(a+bx)\sqrt{1+x^2}}$.

$$\textit{Ans.} \quad \frac{1}{\sqrt{a^2+b^2}} \log\left\{\frac{a+bx}{b - ax + \sqrt{(a^2+b^2)(1+x^2)}}\right\}.$$

This can be integrated either by the method of Art. 13, or by that of Art. 23.

CHAPTER II.

INTEGRATION OF RATIONAL FRACTIONS.

31. **Rational Fractions.**—A fraction whose numerator and denominator are both rational algebraic functions of a variable is called a rational fraction.

Let the expression in question be of the form

$$\frac{ax^m + bx^{m-1} + cx^{m-2} + \&c.}{a'x^n + b'x^{n-1} + c'x^{n-2} + \&c.}$$

in which m and n are positive integers, and $a, b \,.\,.\, a', b', .\,.$ are constants.

In the first place, if the degree of the numerator be greater than, or equal to, that of the denominator, by division we can obtain a quotient together with a new fraction, in which the numerator is of lower degree than the denominator; the former part can be immediately integrated by Art. 3. The integration of the latter part in general comes under the method of Partial Fractions.

32. **Partial Fractions.**—The method of decomposition of a fraction into its partial fractions is usually given in treatises on Algebra; as, however, the process is intimately connected with the integration of a large class of expressions, a short space is devoted to its consideration here.

For brevity, we shall denote the fraction under consideration by $\frac{f(x)}{\phi(x)}$.

Let $a_1, a_2, a_3, \ldots a_n$ denote the roots of $\phi(x)$; then

$$\phi(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n). \qquad (1)$$

There are four cases to be considered, according as we have roots, (1) real and unequal; (2) real and equal; (3) imaginary and unequal; (4) imaginary and equal.

We proceed to discuss each class separately.

33. **Real and Unequal Roots.**—In this case we may assume

$$\frac{f(x)}{\phi(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \frac{A_3}{x - a_3} + \ldots + \frac{A_n}{x - a_n}. \qquad (2)$$

where $A_1, A_2, \ldots . A_n$ are independent of x. For, if the equation be cleared from fractions by multiplying by $\phi(x)$, on equating the coefficients of like powers of x on both sides, we obtain n equations for the determination of the constants $A_1, A_2, \ldots A_n$.

Moreover, since these equations contain A_1, A_2, &c., only in the first degree, they can always be solved: however, since the equations are often too complicated for ready solution, the following method is usually more expeditious.

The equation (2), when cleared from fractions, gives

$$f(x) = A_1(x - a_2)(x - a_3) \ldots (x - a_n) + A_2(x - a_1)(x - a_3) \ldots$$
$$(x - a_n) + \&c. + A_n(x - a_1)(x - a_2) \ldots (x - a_{n-1});$$

and since, by hypothesis, both sides of this equation are identical for all values of x, we may substitute a_1 for x throughout; this gives

$$f(a_1) = A_1(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n).$$

or

$$A_1 = \frac{f(a_1)}{\phi'(a_1)}.$$

In like manner, we have

$$A_2 = \frac{f(a_2)}{\phi'(a_2)}, \; A_3 = \frac{f(a_3)}{\phi'(a_3)}, \; \ldots . \; A_n = \frac{f(a_n)}{\phi'(a_n)}. \qquad (3)$$

Hence, when all the roots are unequal, we have

$$\frac{f(x)}{\phi(x)} = \frac{f(a_1)}{\phi'(a_1)} \frac{1}{x - a_1} + \frac{f(a_2)}{\phi'(a_2)} \frac{1}{x - a_2} + \&c. + \frac{f(a_n)}{\phi'(a_n)} \frac{1}{x - a_n}. \qquad (4)$$

Accordingly, in this case

$$\int \frac{f(x)}{\phi(x)}\,dx = \frac{f(a_1)}{\phi'(a_1)}\log(x - a_1) + \frac{f(a_2)}{\phi'(a_2)}\log(x - a_2) + \&c.$$

$$+ \frac{f(a_n)}{\phi'(a_n)}\log(x - a_n). \qquad (5)$$

The preceding investigation shows that to any root (a), which is *not a multiple root,* corresponds a single term in the integral, viz.,

$$\frac{f(a)}{\phi'(a)}\log(x - a);$$

one which can always be found, whether the remaining roots are known or not; and whether they are real or imaginary.

34. It should also be observed that even when the degree of x in the numerator is greater than, or equal to, that in the denominator, the partial fraction corresponding to any root (a) in the denominator is still of the form found above.

For let

$$\frac{f(x)}{\phi(x)} = Q + \frac{R}{\phi(x)},$$

where Q and R denote the quotient and remainder, and let $\frac{A}{x - a}$ be the partial fraction of $\frac{R}{\phi(x)}$ corresponding to a single root a; then, on multiplying by $\phi(x)$ and substituting a instead of x, it is easily seen, as before, that we get

$$A = \frac{f(a)}{\phi'(a)}.$$

For example, let it be proposed to integrate the expression

$$\frac{x^5\,dx}{x^3 - 2x^2 - 5x + 6}.$$

Here the factors of the denominator are easily seen to be

$$x - 1,\ x + 2, \text{ and } x - 3;$$

accordingly we may assume

$$\frac{x^5}{x^3 - 2x^2 - 5x + 6} = x^2 + ax + \beta + \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}.$$

To find a and β, we equate the coefficients of x^4 and x^3 to zero, after clearing from fractions: this gives immediately, $a = 2$, and $\beta = 9$.

Again, since $\phi(x) = x^3 - 2x^2 - 5x + 6$, we have

$$\phi'(x) = 3x^2 - 4x - 5.$$

Accordingly, substituting 1, -2, and 3, successively for x in the fraction

$$\frac{x^5}{3x^2 - 4x - 5},$$

we get

$$A = -\frac{1}{6}, \quad B = -\frac{32}{15}, \quad C = \frac{243}{10},$$

and hence

$$\frac{x^5}{x^3 - 2x^2 - 5x + 6} = x^2 + 2x + 9 - \frac{1}{6(x - 1)} - \frac{32}{15(x + 2)} + \frac{243}{10(x - 3)}$$

$$\therefore \int \frac{x^5\,dx}{x^3 - 2x^2 - 5x + 6} = \frac{x^3}{3} + x^2 + 9x - \frac{\log(x - 1)}{6}$$

$$- \frac{32}{15}\log(x + 2) + \frac{243}{10}\log(x - 3).$$

35. If the numerator and denominator contain x in even powers only, the process can, in general, be simplified; for, on substituting z for x^2, the fraction becomes of the form

$$\frac{f(z)}{\phi(z)}.$$

Accordingly, whenever the roots of $\phi(z)$ are *real and unequal*, the fraction can be decomposed into partial fractions, and to any root (a) corresponds a fraction of the form

$$\frac{f(a)}{\phi'(a)}\,\frac{1}{z-a}.$$

The corresponding term in the integral of

$$\frac{f(x^2)}{\phi(x^2)}\,dx$$

is obviously represented by

$$\frac{f(a)}{\phi'(a)}\int\frac{dx}{x^2-a}.$$

This is of the form (f) or (h), according as a is a positive or negative root.

The case of imaginary roots in $\phi(z)$ will be considered in a subsequent part of the chapter.

It may be observed that the integrals treated of in Art. 5 are simple cases of the method of partial fractions discussed in this Article.

Examples.

1. $$\int\frac{(2x+3)\,dx}{x^3+x^2-2x}.$$

Here, the factors of the denominator evidently are, x, $x-1$, and $x+2$; we accordingly assume

$$\frac{2x+3}{x^3+x^2-2x}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}.$$

Again, as $\phi(x)=x^3+x^2-2x$, we have $\phi'(x)=3x^2+2x-2$;

$$\therefore\ \frac{f(x)}{\phi'(x)}=\frac{2x+3}{3x^2+2x-2}.$$

Hence, by (3) we have

$$A=-\frac{3}{2},\quad B=\frac{5}{3},\quad C=-\frac{1}{6}.$$

consequently

$$\int \frac{(2x+3)dx}{x^3+x^2-2x} = -\frac{3}{2}\log x + \frac{5}{3}\log(x-1) - \frac{1}{6}\log(x+2).$$

2. $$\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Here

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right).$$

hence the value of the required integral is

$$\frac{1}{(a^2-b^2)}\left\{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) - \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right\}.$$

3. $$\int \frac{xdx}{(x^2+a)(x^2+b)}.$$

Substitute z for x^2, and the transformed integral is

$$\frac{1}{2}\int \frac{dz}{(z+a)(z+b)}.$$

Consequently the value of the required integral is

$$\frac{1}{2(a-b)}\log\left(\frac{x^2+b}{x^2+a}\right).$$

4. $\displaystyle\int \frac{(3x^2-1)dx}{x^2-3x+2}.$ *Ans.* $3x + 11\log(x-2) - 2\log(x-1).$

5. $\displaystyle\int \frac{(x^2-3)\,dx}{x^3-7x+6}.$ $\displaystyle\frac{1}{2}\log(x-1) + \frac{1}{5}\log(x-2) + \frac{3}{10}\log(x-3).$

6. $\displaystyle\int \frac{(2x+1)dx}{x(x+1)(x+2)}.$ $\displaystyle\frac{1}{2}\log x + \log(x+1) - \frac{3}{2}\log(x+2).$

7. $\displaystyle\int \frac{x^2dx}{x^4-x^2-12}.$ $\displaystyle\frac{\sqrt{3}}{7}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{7}\log\left(\frac{x-2}{x+2}\right).$

8. $$\int \frac{dx(a'+b'x^n)}{x^{n+1}(a+bx^n)}.$$

Let $$x^n = \frac{1}{z}.$$

36. **Multiple Real Roots.**—Suppose $\phi(x)$ has r roots each equal to a, then the fraction can be written in the shape

$$\frac{f(x)}{(x-a)^r\,\psi(x)}.$$

In this case we may assume

$$\frac{f(x)}{(x-a)^r\,\psi(x)} = \frac{M_1}{(x-a)^r} + \frac{M_2}{(x-a)^{r-1}} + \ldots + \frac{M_r}{x-a} + \frac{P}{\psi x},$$

where the last term arises from the remaining roots.

For, when the expression is cleared from fractions, it is readily seen that on equating the coefficients of like powers at both sides, we have as many equations as there are unknown quantities, and accordingly the assumption is a legitimate one.

In order to determine the coefficients, M_1, M_2, &c. . . . M_r, clear from fractions, and we get,

$$f(x) = M_1\psi(x) + M_2(x-a)\,\psi(x) + M_3(x-a)^2\,\psi(x) + \&c \;\ldots (6),$$

This gives, when a is substituted for x,

$$f(a) = M_1\,\psi(a), \text{ or } M_1 = \frac{f(a)}{\psi(a)}. \qquad (7)$$

Next, differentiate with respect to x, and substitute a instead of x in the resulting equation, and we get

$$f'(a) = M_1\,\psi'(a) + M_2\,\psi(a)\,; \qquad (8)$$

which determines M_2.

By a second differentiation M_3 can be determined; and so on.

It can be readily seen, that the series of equations thus arrived at may be written as follows—

$$f(a) = M_1\,\psi(a).$$

$$f'(a) = M_1\psi'(a) + 1\,.\,M_2\,\psi(a).$$

$$f''(a) = M_1\psi''(a) + 2\,.\,M_2\,\psi'(a) + 1\,.\,2\,M_3\,\psi(a).$$

$$f'''(a) = M_1\psi'''(a) + 3\,.\,M_2\psi''(a) + 2\,.\,3\,M_3\,\psi'(a) + 1\,.\,2\,.\,3\,.\,M_4\psi(a).$$

$$f^{iv}(a) = M_1\psi^{iv}(a) + 4\,.\,M_2\psi'''(a) + 3\,.\,4\,M_3\psi''(a) + 2\,.\,3\,.\,4\,.\,M_4\psi'(a)$$
$$+ 1\,.\,2\,.\,3\,.\,4\,M_5\psi(a)$$

in which the law of formation is obvious, and the coefficients can be obtained in succession.

The corresponding part of the integral of

$$\frac{f(x)dx}{(x-a)^r\,\psi(x)}$$

evidently is

$$M_r \log(x-a) - \frac{M_{r-1}}{x-a} - \frac{1}{2}\frac{M_{r-2}}{(x-a)^2} - \ldots - \frac{M_1}{(r-1)(x-a)^{r-1}}. \quad (9)$$

If $\phi(x)$ have a second set of multiple roots, the corresponding terms in the integral can be obtained in like manner.

37. **Imaginary Roots.**—The results arrived at in Art. 33 apply to the case of imaginary, as well as to real, roots; however, as the corresponding partial fractions appear in this case under an imaginary form, it is desirable to give an investigation, in which the coefficients are all real.

Suppose $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ to be a pair of conjugate roots in the equation $\phi(x) = 0$; then the corresponding quadratic factor is

$$(x-a)^2 + b^2, \text{ or } x^2 + px + q \text{ (suppose).}$$

We accordingly assume

$$\phi(x) = (x^2 + px + q)\,\psi(x),$$

and, hence

$$\frac{f(x)}{\phi(x)} = \frac{Lx + M}{x^2 + px + q} + \frac{P}{Q};$$

where $\frac{P}{Q}$ represents the portion arising from the remaining roots, and $\frac{Lx + M}{x^2 + px + q}$ is the part arising from the roots $a \pm b\sqrt{-1}$.

Multiplying by $\phi(x)$ we get

$$f(x) = (Lx + M)\,\psi(x) + (x^2 + px + q)\,\frac{P}{Q}\,\psi(x). \quad (7)$$

If in this, $-(px+q)$ be substituted for x^2, the last term disappears; and by repeating the same substitution in the equation

$$f(x) = \psi(x)\,(Lx + M),$$

it ultimately reduces to a simple equation in x; on identifying both sides of this equation, we can determine the values of L and M.

38. In many cases we can determine the coefficients L, M, more expeditiously, either by equating coefficients directly, or else by determining the other partial fractions first, and subtracting their sum from the given fraction.

It will also be found that the determination of many integrals of this class can be much simplified by a transformation to a new variable, or by some other suitable expedient.

Some elementary examples are added for the purpose of illustration.

EXAMPLES.

1. $$\int \frac{x\,dx}{(1+x)(1+x^2)}.$$

Assume

$$\frac{x}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Lx+M}{1+x^2};$$

clearing from fractions, this becomes

$$x = A(1+x^2) + (Lx+M)(1+x).$$

Equate the coefficients, and we get

$$L + A = 0,\ L + M = 1,\ A + M = 0.$$

Hence

$$L = \frac{1}{2},\ M = \frac{1}{2},\ A = -\frac{1}{2};$$

and accordingly

$$\frac{x}{(1+x)(1+x^2)} = -\frac{1}{2}\,\frac{1}{1+x} + \frac{1}{2}\,\frac{1+x}{1+x^2}$$

$$\therefore \int \frac{x\,dx}{(1+x)(1+x^2)} = \frac{1}{4}\log\left|\frac{1+x^2}{(1+x)^2}\right| + \frac{1}{2}\tan^{-1}x.$$

2. $$\int \frac{dx}{1+x^3}.$$

Let

$$\frac{1}{1+x^3} = \frac{A}{1+x} + \frac{Lx+M}{1-x+x^2};$$

consequently $A = \frac{1}{3}$, by formula (3). Substituting and clearing from fractions, we have

$$3 = 1 - x + x^2 + 3(Lx+M)(1+x);$$

hence, dividing by $1+x$, we have

$$2 - x = 3(Lx+M).$$

Consequently,

$$\int \frac{dx}{1+x^3} = \frac{1}{3}\int \frac{dx}{1+x} + \frac{1}{3}\int \frac{(2-x)dx}{1-x+x^2}$$

$$= \frac{1}{3}\log(1+x) - \frac{1}{6}\log(1-x+x^2) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right).$$

3. $\int \frac{dx}{1-x^3}$. *Ans.* $\frac{1}{6}\log\left(\frac{1+x+x^2}{1-2x+x^2}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$

This can be got from the last by changing the sign of x.

4. $$\int \frac{dx}{1-x^6}.$$

In this case we have

$$\frac{1}{1-x^6} = \frac{1}{2}\left(\frac{1}{1-x^3} + \frac{1}{1+x^3}\right).$$

5. $\int \frac{x^7dx}{x^{12}-1}$. *Ans.* $\frac{1}{12}\log\left\{\frac{(x^4-1)^2}{x^8+x^4+1}\right\} + \frac{1}{4\sqrt{3}}\tan^{-1}\left\{\frac{2x^4+1}{\sqrt{3}}\right\}.$

Let $x^4 = z$, and the integral becomes

$$\frac{1}{4}\int \frac{z\,dz}{z^3-1}.$$

6. $$\int \frac{x^2dx}{(x-1)^2(x^2+1)}.$$

Assume

$$\frac{x^2}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Lx+M}{1+x^2}.$$

To find L and M, we clear from fractions, and by Art. 37, the values of L and M are found by making $x^2 = -1$ in the following equation,

$$x^2 = (Lx+M)(x-1)^2.$$

This gives immediately $L = -\frac{1}{2}$, $M = 0$.

Again, by Art. 36, we get immediately, $A = \frac{1}{2}$.

To find B, make $x = 0$ in both sides of our identity, and we get

$$0 = A - B + M, \therefore B = A = \frac{1}{2}.$$

Finally

$$\frac{x^2\,dx}{(x-1)^2(x^2+1)} = \frac{1}{2}\frac{1}{(x-1)^2} + \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{x}{1+x^2}.$$

$$\therefore \int \frac{x^2dx}{(x-1)^2(x^2+1)} = -\frac{1}{2}\frac{1}{x-1} + \frac{1}{2}[\log(x-1) - \frac{1}{4}\log(x^2+1).$$

7. $$\int \frac{dx}{x^8+x^7-x^4-x^3}.$$

Here the denominator is easily seen to be $x^3(x-1)(x+1)^2(x^2+1)$, and the expression becomes

$$\int \frac{dx}{x^3(x-1)(x+1)^2(x^2+1)}.$$

Assume $x = \frac{1}{z}$, and the transformed expression is evidently

$$\int \frac{z^6\,dz}{(z-1)(z+1)^2(z^2+1)}.$$

The quotient is easily seen to be $z-1$, and by the method of Art. 34, we may assume

$$\frac{z^6}{(z-1)(z+1)^2(z^2+1)} = z - 1 + \frac{A}{z-1} + \frac{B}{(z+1)^2} + \frac{C}{z+1} + \frac{Lz+M}{z^2+1}.$$

Hence (Arts. 33, 36) we have

$$A = \frac{1}{8};\ B = -\frac{1}{4}.$$

Next, L and M are found by making $z^2 = -1$, in the equation

$$z^6 = (Lz + M)(z - 1)(z + 1)^2,$$

$$1 = 2(Lz + M)(z + 1) = 2\{Lz^2 + (L + M)z + M\}$$

which gives

$$L + M = 0, \quad M - L = \frac{1}{2}.$$

$$\therefore \; M = \frac{1}{4}, \quad L = -\frac{1}{4}.$$

In order to find the remaining coefficient C, we make $z = 0$, when we get

$$0 = -1 - A + B + C + M, \quad \therefore \; C = \frac{9}{8},$$

hence, we have

$$\frac{z^6}{(z-1)(z+1)^2(z^2+1)} = z - 1 + \frac{1}{8}\frac{1}{z-1} - \frac{1}{4}\frac{1}{(z+1)^2} + \frac{9}{8}\frac{1}{z+1} - \frac{1}{4}\frac{z-1}{z^2+1}.$$

$$\therefore \int \frac{z^6\,dz}{(z-1)(z+1)^2(z^2+1)} = \frac{z^2}{2} - z + \frac{1}{8}\log(z-1) + \frac{1}{4}\frac{1}{z+1}$$

$$+ \frac{9}{8}\log(z+1) - \frac{1}{8}\log(z^2+1) + \frac{1}{4}\tan^{-1} z.$$

Hence,

$$\int \frac{dx}{x^8 + x^7 - x^4 - x^3} = \frac{1}{2x^2} - \frac{1}{x} + \frac{1}{4}\frac{x}{x+1} + \frac{1}{8}\log\frac{1-x}{1+x^2}$$

$$+ \frac{9}{8}\log(x+1) - \log x - \frac{1}{4}\tan^{-1} x.$$

39. **Multiple Imaginary Roots.**—To complete the discussion of the decomposition of the fraction $\frac{f(x)}{\phi(x)}$, suppose the denominator $\phi(x)$ to contain r pairs of equal and imaginary roots, *i.e.*, let the denominator contain a factor of the form $\{(x-a)^2 + b^2\}^r$: and suppose $\phi(x) = \{(x-a)^2 + b^2\}^r \phi_1(x)$.

In this case we assume

$$\frac{f(x)}{\{(x-a)^2+b^2\}^r\phi_1(x)} = \frac{L_1x + M_1}{\{(x-a)^2+b^2\}^r} + \frac{L_2x + M_2}{\{(x-a)^2+b^2\}^{r-1}}$$

$$+ \dots + \frac{L_rx + M_r}{(x-a)^2 + b^2} + \frac{P}{\phi_1(x)}.$$

The remaining partial fractions being obtained from the other roots.

There is no difficulty in seeing that we shall still have as many equations as unknown quantities, $L_1, M_1, L_2, M_2, \ldots$ when the coefficients of like powers of x are equated on both sides.

To determine L_1, M_1, L_2, &c.; let the factor $(x-a)^2+b^2$ be represented by X, and multiply up by X^r, when we get

$$\frac{f(x)}{\phi_1(x)} = L_1 x + M_1 + (L_2 x + M_2)X + \ldots + (L_r x + M_r)X^{r-1} + \frac{PX^r}{\phi_1(x)} \qquad (11)$$

The coefficients L_1 and M_1 are determined as in Art. 37. To find L_2 and M_2; differentiate with respect to x, and substitute $a+b\sqrt{-1}$ for x in the result, when it becomes

$$\frac{d}{dx}\left[\frac{f(x)}{\phi_1(x)}\right]_0 = L_1 + (L_2 x_0 + M_2)\,2(x_0 - a).$$

where $x_0 = a + b\sqrt{-1}$.

Hence, equating real and imaginary parts, we get two equations for the determination of L_2 and M_2. By a second differentiation, L_3 and M_3 can be determined, and so on.

It is unnecessary to go more into detail, as sufficient has been stated to show that the decomposition into partial fractions is possible in all cases, when the roots of $\phi(x)=0$ are known.

The practical application is often simplified by transformation to a new variable.

40. The preceding investigation shows that the integration of rational fractions is in all cases reducible to that of one or more fractions of the following forms

$$\frac{dx}{x-a},\quad \frac{dx}{(x-a)^r},\quad \frac{(A+Bx)dx}{(x-a)^2+b^2},\quad \frac{(Lx+M)dx}{\{(x-a)^2+b^2\}^r}.$$

The methods of integrating the three first forms have been given already. We proceed to show the mode of dealing with the last.

41. In the first place it can be divided into two others,

$$\frac{L(x-a)\,dx}{\{(x-a)^2+b^2\}^r}+\frac{(La+M)dx}{\{(x-a)^2+b^2\}^r}.$$

The integral of the first part is evidently

$$\frac{-L}{2(r-1)\,\{(x-a)^2+b^2\}^{r-1}}.$$

To determine the integral of the other part, we substitute z for $x-a$, and, omitting the constant coefficient, it becomes

$$\int\frac{dz}{(z^2+b^2)^r}.$$

Again,

$$\int\frac{dz}{(z^2+b^2)^r}=\frac{1}{b^2}\int\frac{(z^2+b^2-z^2)dz}{(z^2+b^2)^r}=\frac{1}{b^2}\int\frac{dz}{(z^2+b^2)^{r-1}}-\frac{1}{b^2}\int\frac{z^2\,dz}{(z^2+b^2)^r},$$

But we get by integration by parts,

$$\int\frac{z^2\,dz}{(z^2+b^2)^r}=\int z\cdot\frac{z\,dz}{(z^2+b^2)^r}=-\frac{1}{2(r-1)}\int z\,d\left(\frac{1}{(z^2+b^2)^{r-1}}\right).$$

$$=-\frac{z}{2(r-1)\,(z^2+b^2)^{r-1}}+\frac{1}{2(r-1)}\int\frac{dz}{(z^2+b^2)^{r-1}}.$$

Substituting in the preceding, we obtain

$$\int\frac{dz}{(z^2+b^2)^r}=\frac{2r-3}{2(r-1)b^2}\int\frac{dz}{(z^2+b^2)^{r-1}}+\frac{z}{2(r-1)b^2\,(z^2+b^2)^{r-1}}.\quad(12)$$

This formula reduces the integral to another of the same shape, in which the exponent r is replaced by $r-1$. By successive repetitions of this formula the integral can be reduced to depend on that of $\frac{dz}{z^2+b^2}$.

The preceding is a case of the method of integration by *successive reduction*, referred to in Art. 19. Other examples of this method will be found in the next Chapter.

The preceding integral can often be found more expeditiously by the following transformation. Substitute $b \tan \theta$ for z, and the expression $\frac{dz}{(z^2 + b^2)^r}$ becomes, obviously

$$\frac{1}{b^{2r-1}} \int \cos^{2r-2} \theta \, d\theta.$$

The discussion of this class of integrals will be found in the next Chapter.

42. We shall next return to the integration of $\frac{f(x^2)\,dx}{\phi(x^2)}$ which has been already considered in Art. 35, in the case where the roots of $\phi(z)$ are real. To a pair of imaginary roots, $a \pm b\sqrt{-1}$, corresponds a partial fraction of the form

$$\frac{(Ax^2 + B)\,dx}{(x^2 - a)^2 + b^2}, \text{ or } \frac{(Ax^2 + B)\,dx}{x^4 - 2ax^2 + c^2},$$

where $c^2 = a^2 + b^2$.

In order to integrate this, we assume $a = c \cos 2\phi$, when the fraction becomes

$$\frac{(Ax^2 + B)\,dx}{x^4 - 2x^2 c \cos 2\phi + c^2}.$$

The quadratic factors of the denominator are easily seen to be

$$x^2 - 2x\sqrt{c} \cos \phi + c, \text{ and } x^2 + 2x\sqrt{c} \cos \phi + c.$$

Accordingly, we assume

$$\frac{Ax^2 + B}{x^4 - 2x^2 c \cos 2\phi + c^2} = \frac{Lx + M}{x^2 - 2x\sqrt{c} \cos \phi + c} + \frac{L'x + M'}{x^2 + 2x\sqrt{c} \cos \phi + c}:$$

hence, it can be seen without difficulty, that

$$L = -L' = \frac{Ac - B}{8\,c^{\frac{3}{2}} \cos \phi}, \quad M = M' = \frac{B}{2c},$$

and after a few easy transformations, we find

$$\int \frac{(Ax^2 + B)\,dx}{x^4 - 2x^2 c \cos 2\phi + c^2} = \frac{Ac - B}{8 \cos\phi\, c^{\frac{3}{2}}} \log\left(\frac{x^2 - 2x\sqrt{c}\cos\phi + c}{x^2 + 2x\sqrt{c}\cos\phi + c}\right)$$

$$+ \frac{Ac + B}{4 \sin\phi\, c^{\frac{3}{2}}} \tan^{-1}\left(\frac{2x\sqrt{c}\sin\phi}{c^2 - x^2}\right).$$

43. **Integration of** $\dfrac{dx}{(x - a)^m (x - b)^n}$.

This expression can be easily transformed into a shape which is immediately integrable, by the following substitution:—

Assume $x - a = (x - b)\,z$, then

$$x = \frac{a - bz}{1 - z}, \;\therefore\; x - a = \frac{(a - b)\,z}{1 - z},\; x - b = \frac{a - b}{1 - z},\; dx = \frac{(a - b)\,dz}{(1 - z)^2}.$$

and the expression transforms into

$$\frac{(1 - z)^{m+n-2}\,dz}{(a - b)^{m+n-1} z^m}.$$

Expand the numerator by the binomial theorem, and the integral can be immediately obtained.

For example, take the integral

$$\frac{dx}{(x - a)^2 (x - b)^3}$$

Here the transformed expression is

$$\int \frac{(1 - z)^3\,dz}{(a - b)^4 z^2},$$

or

$$\frac{1}{(a - b)^4} \int \left(\frac{1}{z^2} - \frac{3}{z} + 3 - z\right) dz = \frac{-1}{(a - b)^4} \left\{\frac{z^2}{2} - 3z + 3 \log z + \frac{1}{z}\right\}.$$

Substituting $\dfrac{x - a}{x - b}$ for z, the integral can be expressed in terms of x.

44. **Integration of** $\frac{x^{2m+1}\,dx}{(a + cx^2)^n}$,

where m and n are integers.

Let $a + cx^2 = z$, and the expression becomes

$$\frac{(z - a)^m\,dz}{2c^{m+1}\,z^n};$$

a form which is immediately integrable by aid of the Binomial Theorem.

It is evident that the expression is made integrable by the same transformation when n is a fractional or negative index.

It may be also observed that the more general expression $\frac{f(x^2)\,xdx}{(a + cx^2)^n}$ can be integrated by the same transformation, where $f(x^2)$ denotes an integer algebraic function of x^2.

EXAMPLES.

1. $\int \frac{x^5\,dx}{(a^2 - x^2)^2}$. *Ans.* $\frac{a^4}{2(a^2 - x^2)} + \frac{x^2}{2} + a^2 \log (a^2 - x^2)$.

2. $\int \frac{x^3\,dx}{(a + cx^2)^4}$. $-\frac{1}{4c^2(a + cx^2)^2} + \frac{a}{6c^2(a + cx^2)^3}$.

3. $\int \frac{x^5\,dx}{(1 + x^2)^3}$. $\frac{1}{x^2 + 1} - \frac{1}{4(x^2 + 1)^2} + \frac{1}{2}\log (x^2 + 1)$.

45. **Integration of** $\frac{dx}{x^n - 1}$.

where n is a positive integer.

Suppose a an imaginary root of $x^n - 1 = 0$, and it is evident that a^{-1} is the conjugate root: also by (3), the partial fraction corresponding to the root a is

$$\frac{1}{na^{n-1}(x - a)}, \text{ or } \frac{a}{n(x - a)}.$$

If to this the fraction arising from the root a^{-1} be added, we get

$$\frac{1}{n}\left\{\frac{a}{x-a}+\frac{a^{-1}}{x-a^{-1}}\right\} \text{ or } \frac{1}{n}\left\{\frac{x(a+a^{-1})-2}{x^2-(a+a^{-1})x+1}\right\}.$$

But, by the theory of equations, a is of the form

$$\cos\frac{2k\pi}{n}+\sqrt{-1}\sin\frac{2k\pi}{n},$$

where k is an integer,

$$\therefore\ a+a^{-1}=2\cos\frac{2k\pi}{n}.$$

Hence, if θ be substituted for $\frac{2k\pi}{n}$ the preceding fraction becomes

$$\frac{2}{n}\cdot\frac{x\cos\theta-1}{x^2-2x\cos\theta+1}.$$

The integral of this, by Art. 7, is

$$\frac{\cos\theta}{n}\log\left(1-2x\cos\theta+x^2\right)-\frac{2\sin\theta}{n}\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right).$$

There are two cases to be considered, according as n is even or odd.

(1.) Let $n=2r$: in this case the equation $x^{2r}-1=0$ has two real roots, viz. $+1$ and -1; and it is easily seen that

$$\int\frac{dx}{x^{2r}-1}=\frac{1}{2r}\log\frac{x-1}{x+1}+\frac{1}{2r}\sum\cos\frac{k\pi}{r}\log\left(1-2x\cos\frac{k\pi}{r}+x^2\right)$$

$$-\frac{1}{r}\sum\sin\frac{k\pi}{r}\tan^{-1}\left(\frac{x-\cos\frac{k\pi}{r}}{\sin\frac{k\pi}{r}}\right), \qquad (13)$$

where the summation represented by Σ extends to all integral values of k from 1 to $r - 1$.

(2.) Let $n = 2r + 1$, and we obtain

$$\int \frac{dx}{x^{2r+1} - 1} = \frac{\log(x-1)}{2r+1} + \frac{1}{2r+1}\Sigma \cos\frac{2k\pi}{2r+1}\log\left(1 - 2x\cos\frac{2k\pi}{2r+1} + x^2\right)$$

$$- \frac{2}{2r+1}\Sigma \sin\frac{2k\pi}{2r+1}\tan^{-1}\left(\frac{x - \cos\dfrac{2k\pi}{2r+1}}{\sin\dfrac{2k\pi}{2r+1}}\right) \quad (14)$$

where the summation represented by Σ extends to all integral values of k from 1 up to r.

46. **Integration of** $\dfrac{x^{m-1}dx}{x^n - 1}$ where m is less than n.

As before, let a be a root, and the corresponding partial fraction is $\dfrac{a^{m-1}}{na^{n-1}(x-a)}$ or $\dfrac{a^m}{n(x-a)}$; hence the partial fraction arising from the complementary roots, a and a^{-1}, is

$$\frac{1}{n}\left(\frac{a^m}{x-a} + \frac{a^{-m}}{x - a^{-1}}\right) = \frac{1}{n}\cdot\frac{x(a^m + a^{-m}) - (a^{m-1} + a^{-(m-1)})}{x^2 - (a + a^{-1})x + 1}$$

$$= \frac{2}{n}\,\frac{x\cos m\theta - \cos(m-1)\theta}{x^2 - 2x\cos\theta + 1},$$

where θ is of the same form as before.

The corresponding term in the proposed integral is easily seen, by Art. 7, to be

$$\frac{1}{n}\left\{\cos m\theta \log(x^2 - 2x\cos\theta + 1) - 2\sin m\theta \tan^{-1}\frac{x - \cos\theta}{\sin\theta}\right\}. \quad (15)$$

By giving to k all values from 1 to $\dfrac{n}{2} - 1$, when n is even; and from 1 to $\dfrac{n-1}{2}$ when n is odd, the integral required can be written down as in the preceding Article.

EXAMPLES.

1. $\int \frac{dx}{x^2 + 6x + 8}$. *Ans.* $\frac{1}{2} \log \left(\frac{x + 2}{x + 4} \right)$.

2. $\int \frac{dx.}{x^4 - 1}$ $\frac{1}{4} \log \left(\frac{x - 1}{x + 1} \right) - \frac{1}{2} \tan^{-1} x$.

3. $\int \frac{(A + Bx^2)dx}{x(a + bx^2)}$. $\frac{A}{a} \log x + \frac{Ba - Ab}{2ab} \log (a + bx^2)$.

4. $\int \frac{x^2 dx}{x^4 + x^2 - 2}$. $\frac{1}{6} \log \frac{x - 1}{x + 1} + \frac{1}{3\sqrt{2}} \tan^{-1} \left(\frac{x^2}{\sqrt{2}} \right)$.

5. $\int \frac{dx}{x^4 + 1}$ $\frac{1}{4\sqrt{2}} \log \left(\frac{x^2 + \sqrt{2} + 1}{x^2 - x\sqrt{2} + 1} \right) + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x\sqrt{2}}{1 - x^2} \right)$

6. $\int \frac{(2x - 5)\, dx}{(x + 3)(x + 1)^2}$. $\frac{7}{2(x + 1)} + \frac{11}{4} \log \left(\frac{x + 1}{x + 3} \right)$.

7. $\int \frac{dx}{x(a + bx^2)^2}$. $\frac{1}{2a(a + bx^2)} + \frac{1}{2a^2} \log \left(\frac{x^2}{a + bx^2} \right)$.

8. $\int \frac{dx}{x(a + bx^n)^2}$. $\frac{1}{na^2} \log \left(\frac{x^n}{a + bx^n} \right) + \frac{1}{na(a + bx^n)}$.

9. $\int \frac{dx}{x(a + bx^n)^r}$.

Let $a + bx^n = x^n z$, and the transformed expression is $-\frac{(z - b)^{r-1} dz}{na^r z^r}$.

10. $\int \frac{xdx}{x^3 + x^2 + x + 1}$. $\frac{1}{4} \log (x^2 + 1) - \frac{1}{2} \log (x + 1) + \frac{1}{2} \tan^{-1} x$.

11. $\int \frac{dx}{x^4 + 4x^3 + 5x^2 + 4x + 4}$. $\frac{4}{25} \log (x + 2) - \frac{2}{25} \log (x^2 + 1) + \frac{3}{25} \tan^{-1} x$.

12. Apply the method of Art. 43 to the integration of $\frac{dx}{(1 - x^2)^n}$.

The transformed expression is $-\frac{(1 + z)^{2n-2} dz}{2^{2n-1} z^n}$.

13. $\int \frac{x^2 dx}{(1 - x^2)^3}$.

14. By a similar transformation, prove that

$$\int \frac{dx}{x^n(1-x)^m} = -\int \frac{(1+z)^{m+n-2}dz}{z^n}.$$

15. $\displaystyle\int \frac{dx}{\sin x(a+b\cos x)}.$ $\displaystyle\frac{1}{a-b}\log\sin\frac{x}{2} - \frac{1}{a+b}\log\cos\frac{x}{2}$

$$+ \frac{b}{a^2-b^2}\log(a+b\cos x).$$

Multiply by $\sin x$, substitute u for $\cos x$, and the integral becomes

$$\int \frac{-du}{(1-u^2)(a+bu)}.$$

16. $\displaystyle\int \frac{dx}{3\sin x + \sin 2x}.$ $\displaystyle\log\sin\frac{x}{2} - \frac{1}{5}\log\cos\frac{x}{2} + \frac{2}{5}\log(3+2\cos x).$

17. $\displaystyle\int \frac{(1-x^2)dx}{x(1+x^2+x^4)}.$ $\displaystyle\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2+x^2}{x^2\sqrt{3}}\right) - \frac{1}{4}\log\left(\frac{x^4+x^2+1}{x^4}\right).$

Let $x^2 = \frac{1}{z}$, &c.

18. Prove that

$$\int \frac{dx}{1+x^{2n}} = -\frac{1}{2n}\Sigma\cos\frac{(k-1)\pi}{2n}\log\left(1-2x\cos\frac{(2k-1)\pi}{2n}+x^2\right)$$

$$+\frac{1}{n}\Sigma\sin\frac{(2k-1)\pi}{2n}\tan^{-1}\left\{\frac{x-\cos\frac{(2k-1)\pi}{2n}}{\sin\frac{(2k-1)\pi}{2n}}\right\};$$

where k extends through all integral values from 1 to n, inclusive.

19. $$\int \frac{dx}{1+x^{2n+1}} = \frac{\log(1+x)}{2n+1} - \frac{1}{2n+1}\Sigma\cos\frac{(2k-1)\pi}{2n+1}\log\left(1-2x\cos\frac{(2k-1)\pi}{2n+1}+x^2\right)$$

$$+\frac{2}{2n+1}\Sigma\sin\frac{(2k-1)\pi}{2n+1}\tan^{-1}\left\{\frac{x-\cos\frac{(2k-1)\pi}{2n+1}}{\sin\frac{(2k-1)\pi}{2n+1}}\right\}.$$

When k assumes all integral values from 1 to n inclusive.

CHAPTER III.

INTEGRATION BY SUCCESSIVE REDUCTION.

48. **Cases in which $\sin^m\theta\cos^n\theta\,d\theta$ is immediately Integrable.**— We shall commence this Chapter* with the discussion of the integral

$$\int \sin^m\theta \cos^n\theta\, d\theta;$$

to which form it will be seen that a number of other expressions are readily reducible.

In the first place it is easily seen that whenever *either m or n is an odd positive integer*, the expression $\sin^m\theta\cos^n\theta d\theta$ can be immediately integrated.

For, if $n = 2r + 1$, the integral becomes

$$\int \sin^m\theta \cos^{2r+1}\theta\, d\theta, \text{ or } \int \sin^m\theta (\cos^2\theta)^r\, d(\sin\theta);$$

if we assume $x = \sin\theta$, the integral transforms into

$$\int x^m (1 - x^2)^r\, dx, \qquad (1)$$

and as, by hypothesis, r is a positive integer, $(1 - x^2)^r$ can be expanded by the Binomial Theorem in a finite number of terms, each of which can be integrated separately. In like manner, if the index of $\sin\theta$ be an odd integer, we assume $x = \cos\theta$, &c.

A few examples are added for the purpose of making the student familiar with this principle.

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[*] It may be observed that a large number of the Integrals discussed in this Chapter do not require the method of successive Reduction: however, since other Integrals of the same form require this method, it was not considered advisable to separate the discussion into distinct chapters.

EXAMPLES.

1. $\int \sin^3\theta\,d\theta.$ *Ans.* $\dfrac{\cos^3\theta}{3} - \cos\theta.$

2. $\int \cos^5\theta\,d\theta$ $\quad \sin\theta - \dfrac{2}{3}\sin^3\theta + \dfrac{\sin^5\theta}{5}.$

3. $\int \sin^3\theta\cos^7\theta\,d\theta.$ $\quad \dfrac{\cos^{10}\theta}{10} - \dfrac{\cos^8\theta}{8}.$

4. $\int \dfrac{\sin^5\theta\,d\theta}{\cos^2\theta}.$ $\quad \dfrac{1}{\cos\theta} + 2\cos\theta - \dfrac{\cos^3\theta}{3}.$

5. $\int \sqrt{\sin\theta}\cos^3\theta\,d\theta.$ $\quad \dfrac{2\sin^{\frac{3}{2}}\theta}{3} - \dfrac{2\sin^{\frac{7}{2}}\theta}{7}.$

6. $\int \dfrac{\sin^3\theta\,d\theta}{\sqrt{\cos\theta}}.$ $\quad \dfrac{2\cos^{\frac{5}{2}}\theta}{5} - 2\cos^{\frac{1}{2}}\theta.$

7. $\int \dfrac{\sin^3\theta\,d\theta}{\cos^{\frac{p}{q}}\theta}.$ $\quad \dfrac{q}{p-q}\cos^{1-\frac{p}{q}}\theta + \dfrac{q}{3q-p}\cos^{3-\frac{p}{q}}\theta.$

49. (2). Again, whenever $m + n$ *is an even negative integer*, the expression $\sin^m\theta\cos^n\theta\,d\theta$ can be readily integrated.

For if we assume $x = \tan\theta$, we have

$$\cos\theta = \frac{1}{\sqrt{1+x^2}},\quad \sin\theta = \frac{x}{\sqrt{1+x^2}},\quad \text{and } d\theta = \frac{dx}{1+x^2},$$

and the expression transforms into

$$\frac{x^m\,dx}{(1+x^2)^{\frac{m+n}{2}+1}}.$$

Hence, if $m + n = -2r$, this becomes

$$x^m(1+x^2)^{r-1}\,dx,$$

a form which is immediately integrable.

Take for example $\displaystyle\int\frac{\sin^2\theta d\theta}{\cos^6\theta}$.

Let $x = \tan\theta$, and we get

$$\int x^2(1+x^2)\,dx, \text{ or } \frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}.$$

Next, to find $\displaystyle\int\frac{d\theta}{\sin\theta\cos^5\theta}$.

Making the same substitution, we obtain

$$\int\frac{(1+x^2)^2\,dx}{x}.$$

Hence, the value of the proposed integral is

$$\frac{\tan^4\theta}{4}+\tan^2\theta+\log(\tan\theta).$$

Again, to find $\displaystyle\int\frac{d\theta}{\sin^{\frac{3}{2}}\theta\cos^{\frac{5}{2}}\theta}$.

Here the transformed expression is $\dfrac{(1+x^2)\,dx}{x^{\frac{3}{2}}}$, and accordingly the value of the proposed integral is

$$\frac{2}{3}\tan^{\frac{3}{2}}\theta-\frac{2}{\tan^{\frac{1}{2}}\theta}.$$

In many cases it is more convenient to assume $x = \cot\theta$. For example, to find $\displaystyle\int\frac{d\theta}{\sin^4\theta}$.

Since $d(\cot\theta) = -\dfrac{d\theta}{\sin^2\theta}$, if $\cot\theta = x$, the transformed integral is

$$-\int(1+x^2)\,dx, \text{ or } -\cot\theta-\frac{\cot^3\theta}{3}.$$

The following examples are added for illustration.

EXAMPLES.

1. $\displaystyle\int \frac{\sin^3\theta\, d\theta}{\cos^5\theta}.$ *Ans.* $\displaystyle\frac{\tan^4\theta}{4}.$

2. $\displaystyle\int \frac{d\theta}{\cos^6\theta}.$ $\displaystyle\tan\theta + \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5}.$

3. $\displaystyle\int \frac{d\theta}{\sin\theta\cos^3\theta}.$ $\displaystyle\frac{\tan^2\theta}{2} + \log(\tan\theta).$

4. $\displaystyle\int \frac{\sin^{\frac{1}{2}}\theta\, d\theta}{\cos^{\frac{5}{2}}\theta}.$ $\displaystyle\frac{2}{3}\tan^{\frac{3}{2}}\theta.$

5. $\displaystyle\int \frac{d\theta}{\sin^4\theta\cos^4\theta}.$ $\displaystyle-8\cot 2\theta - \frac{8}{3}\cot^3 2\theta.$

6. $\displaystyle\int \frac{d\theta}{\sin^{\frac{1}{2}}\theta\cos^{\frac{7}{2}}\theta}.$ $\displaystyle 2\tan^{\frac{1}{2}}\theta\left(1 + \frac{\tan^2\theta}{5}\right).$

When neither of the preceding methods is applicable, the integration of the expression $\sin^m\theta \cos^n\theta\, d\theta$ can be obtained only by aid of successive reduction.

We proceed to establish the formulæ of reduction suitable to this case.

50. **Formulæ of Reduction for** $\sin^m\theta \cos^n\theta\, d\theta$.

$$\int \sin^m\theta\cos^n\theta\, d\theta = \int \cos^{n-1}\theta\, \sin^m\theta\, d(\sin\theta):$$

consequently, if we assume

$$u = \cos^{n-1}\theta, \quad v = \frac{\sin^{m+1}\theta}{m+1},$$

the formula for integration by parts (Art. 21), gives

$$\int \sin^m\theta\cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} + \frac{n-1}{m+1}\int \sin^{m+2}\theta\cos^{n-2}\theta\, d\theta. \quad (2)$$

In like manner, if the integral be written in the form

$$-\int \sin^{m-1}\theta \cos^n\theta d(\cos\theta),$$

we obtain

$$\int \sin^m\theta \cos^n\theta d\theta = \frac{m-1}{n+1}\int \sin^{m-2}\theta \cos^{n+2}\theta\, d\theta - \frac{\sin^{m-1}\theta \cos^{n+1}\theta}{n+1}. \quad (3)$$

It may be observed that this latter formula can be derived from (2) by substituting $90^\circ - \phi$ for θ, and interchanging the letters m and n, in it.

51. **Case of one Positive and one Negative Index.**—The results in (2) and (3) hold whether m or n be positive or negative: accordingly, let one of them be negative (n suppose), and on changing n into $-n$, formula (3) becomes

$$\int \frac{\sin^m\theta}{\cos^n\theta} d\theta = \frac{\sin^{m-1}\theta}{(n-1)\cos^{n-1}\theta} - \frac{m-1}{n-1}\int \frac{\sin^{m-2}\theta}{\cos^{n-2}\theta} d\theta. \quad (A)$$

in which m and n are supposed to have positive* signs.

By this formula the integral of $\frac{\sin^m\theta}{\cos^n\theta} d\theta$ is made to depend on another in which the indices of $\sin\theta$ and $\cos\theta$ are each diminished by two. The same method is applicable to the new integral and so on.

If m be an odd integer, the expression is integrable immediately by Art. 48. If m be even, and n even and greater than m, the method of Art. 49 is applicable; if $m = n$, the expression becomes $\int \tan^m\theta\, d\theta$; which will be treated subsequently; if $n < m$, the integral reduces to that of $\sin^{m-n}\theta d\theta$. Again, if n be odd, and $> m$, the integral reduces to $\int \frac{d\theta}{\cos^{n-m}\theta}$;

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* The formulæ of reduction employed in practice are indicated by the capital letters, *A*, *B*, &c.: and in them the indices m and n are supposed to have always positive signs. By this means the formulæ will be more easily apprehended and applied by the student.

and if $n < m$, it reduces to $\int \frac{\sin^{m-n+1}\theta\, d\theta}{\cos\theta}$. The mode of finding these latter integrals will be considered subsequently.

Again, if the index of sin θ be negative, we get, by changing the sign of m in (2),

$$\int \frac{\cos^n\theta}{\sin^m\theta}\, d\theta = -\frac{\cos^{n-1}\theta}{(m-1)\sin^{m-1}\theta} - \frac{n-1}{m-1}\int \frac{\cos^{n-2}\theta}{\sin^{m-2}\theta}\, d\theta. \quad (B)$$

We shall next consider the case where the indices are both positive.

52. **Indices both Positive.**—If $\sin^m\theta\,(1-\cos^2\theta)$ be written instead of $\sin^{m+2}\theta$ in formula (2), it becomes

$$\int \sin^m\theta \cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta \sin^{m+1}\theta}{m+1} +$$

$$\frac{n-1}{m+1}\int \sin^m\theta\,(\cos^{n-2}\theta - \cos^n\theta)\, d\theta = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+1} +$$

$$\frac{n-1}{m+1}\int \sin^m\theta\cos^{n-2}\theta\, d\theta - \frac{n-1}{m+1}\int \sin^m\theta\cos^n\theta\, d\theta;$$

hence, transposing the latter integral to the other side, and dividing by $\frac{m+n}{m+1}$, we get

$$\int \sin^m\theta\cos^n\theta\, d\theta = \frac{\cos^{n-1}\theta\sin^{m+1}\theta}{m+n} + \frac{n-1}{m+n}\int \sin^m\theta\cos^{n-2}\theta\, d\theta. \quad (C)$$

In like manner, from (3), we get

$$\int \sin^m\theta\cos^n\theta\, d\theta = \frac{m-1}{m+n}\int \sin^{m-2}\theta\cos^n\theta\, d\theta - \frac{\sin^{m-1}\theta\cos^{n+1}\theta}{m+n} \quad (D)$$

By aid of these formulæ the integral of $\sin^m\theta\cos^n\theta\, d\theta$ is made to depend on another in which the index of either sin θ, or of cos θ, is reduced by two. By successive application of these formulæ, the complete integral can always be found when the indices are *integers*.

53. **Formulæ of Reduction for $\sin^n\theta\,d\theta$ and $\cos^n\theta\,d\theta$.** These integrals are evidently cases of the general formulæ (C) and (D); however they are so frequently employed that we give the formulæ of reduction separately in their case,

$$\int\cos^n\theta\,d\theta = \frac{\sin\theta\cos^{n-1}\theta}{n} + \frac{n-1}{n}\int\cos^{n-2}\theta\,d\theta. \qquad (4)$$

$$\int\sin^n\theta\,d\theta = -\frac{\cos\theta\sin^{n-1}\theta}{n} + \frac{n-1}{n}\int\sin^{n-2}\theta\,d\theta. \qquad (5)$$

The former gives, when n is even,

$$\int\cos^n\theta\,d\theta = \frac{\sin\theta}{n}\left(\cos^{n-1}\theta + \frac{n-1}{n-2}\cos^{n-3}\theta\right.$$
$$\left. + \frac{(n-1)(n-3)}{(n-2)(n-4)}\cos^{n-5}\theta + \&c.\right) +$$
$$+ \frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 2}\,\theta. \quad \ldots \quad (6)$$

EXAMPLES.

1. $\int\sin^4\theta\,d\theta$. *Ans.* $-\cos\theta\left(\frac{\sin^3\theta}{4}+\frac{3}{8}\sin\theta\right)+\frac{3}{8}\theta$.

2. $\int\cos^2\theta\sin^4\theta\,d\theta$. $\left(\frac{\sin^5\theta}{6}-\frac{\sin^3\theta}{24}-\frac{\sin\theta}{16}\right)\cos\theta+\frac{\theta}{16}$.

3. $\int\cos^6\theta\,d\theta$.

54. **Indices both negative.**—It remains to consider the case where the indices of $\sin\theta$ and $\cos\theta$ are both negative.

Writing $-m$ and $-n$ instead of m and n in formula (C), it becomes

$$\int\frac{d\theta}{\sin^m\theta\cos^n\theta} = \frac{-1}{(m+n)\cos^{n+1}\theta\sin^{m-1}\theta} + \frac{n+1}{m+n}\int\frac{d\theta}{\sin^m\theta\cos^{n+2}\theta},$$

or, transposing and multiplying by $\frac{m+n}{n+1}$,

$$\int \frac{d\theta}{\sin^m\theta\cos^{n+2}\theta} = \frac{1}{(n+1)\cos^{n+1}\theta\sin^{m-1}\theta} + \frac{m+n}{n+1}\int\frac{d\theta}{\sin^m\theta\cos^n\theta}.$$

Again, if we substitute n for $n+2$ in this, it becomes

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \frac{1}{(n-1)\cos^{n-1}\theta\sin^{m-1}\theta} + \frac{m+n-2}{n-1}\int\frac{d\theta}{\sin^m\theta\cos^{n-2}\theta}. \qquad (E)$$

Making a like transformation* in formula D, it becomes

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \frac{-1}{(m-1)\sin^{m-1}\theta\cos^{n-1}\theta} + \frac{m+n-2}{m-1}\int\frac{d\theta}{\sin^{m-2}\theta\cos^n\theta}. \qquad (F)$$

In each of these, one of the indices is reduced by two degrees, and consequently by successive applications of the formulæ, the integrals are reducible ultimately to those of one or other of the forms $\frac{d\theta}{\cos\theta}$ or $\frac{d\theta}{\sin\theta}$: these have been already integrated in Art. 17.

The formulæ of reduction for $\frac{d\theta}{\sin^n\theta}$ and $\frac{d\theta}{\cos^n\theta}$, are so important that they are added independently, as follows :—

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* It may observed that formulæ (*B*), (*D*) and (*F*) can be immediately obtained from (*A*), (*C*) and (*E*), by interchanging the letters m and n, and substituting $\frac{\pi}{2} - \phi$ instead of θ. For, in this case, $\sin\theta$, $\cos\theta$ and $d\theta$, transform into $\cos\phi$, $\sin\phi$ and $-d\phi$, respectively, &c.

$$\int \frac{d\theta}{\cos^n\theta} = \frac{\sin\theta}{(n-1)\cos^{n-1}\theta} + \frac{n-2}{n-1}\int\frac{d\theta}{\cos^{n-2}\theta}. \tag{6}$$

$$\int \frac{d\theta}{\sin^n\theta} = \frac{-\cos\theta}{(n-1)\sin^{n-1}\theta} + \frac{n-2}{n-1}\int\frac{d\theta}{\sin^{n-2}\theta}. \tag{7}$$

It may be here observed that since $\sin^2\theta + \cos^2\theta = 1$, we have immediately

$$\int \frac{d\theta}{\sin^m\theta\cos^n\theta} = \int\frac{d\theta}{\sin^{m-2}\theta\cos^n\theta} + \int\frac{d\theta}{\sin^m\theta\cos^{n-2}\theta}; \tag{8}$$

and a similar process is applicable to the latter integrals. This method is often useful in elementary cases.

EXAMPLES.

1. $\displaystyle\int\frac{d\theta}{\sin\theta\cos^2\theta} = \int\frac{\sin\theta\,d\theta}{\cos^2\theta} + \int\frac{d\theta}{\sin\theta} = \frac{1}{\cos\theta} + \log\tan\frac{\theta}{2}.$

2. $\displaystyle\int\frac{d\theta}{\sin\theta\cos^4\theta} = \int\frac{\sin\theta\,d\theta}{\cos^4\theta} + \int\frac{d\theta}{\sin\theta\cos^2\theta},$

and is accordingly immediately integrated by the last.

3. $\displaystyle\int\frac{d\theta}{\sin^3\theta\cos^2\theta}.$ *Ans.* $\displaystyle\frac{1}{3\sin^3\theta} - \frac{1}{\sin\theta} + \log\left\{\tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)\right\}$

4. $\displaystyle\int\frac{d\theta}{\sin^3\theta}.$ $\displaystyle-\frac{\cos\theta}{2\sin^2\theta} + \frac{1}{2}\log\tan\frac{\theta}{2}.$

55. **Integration of** $\tan^n\theta\,d\theta$ **and** $\dfrac{d\theta}{\tan^n\theta}$.

These integrals may be regarded as cases of the preceding; they can, however, be arrived at in a simpler manner, as follows:—

Since $\tan^2\theta = \sec^2\theta - 1$, we have

$$\int\tan^n\theta\,d\theta = \int\tan^{n-2}\theta\,(\sec^2\theta - 1)\,d\theta = \int\tan^{n-2}\theta\,d(\tan\theta)$$

$$-\int\tan^{n-2}\theta\,d\theta = \frac{\tan^{n-1}\theta}{n-1} - \int\tan^{n-2}\theta\,d\theta. \tag{9}$$

By aid of this formula we have, at once

$$\int \tan^n\theta\, d\theta = \frac{\tan^{n-1}\theta}{n-1} - \frac{\tan^{n-3}\theta}{n-3} + \frac{\tan^{n-5}\theta}{n-5} - \&c. \qquad (10)$$

(1.) If $n = 2r + 1$, the last term is easily seen to be $(-1)^r \log(\cos\theta)$.

(2). If $n = 2r$, the two last terms may be represented by $(-1)^{r+1}(\tan\theta - \theta)$.

In a similar manner we have

$$\int\frac{d\theta}{\tan^n\theta} = \int\frac{\sec^2\theta d\theta}{\tan^n\theta} - \int\frac{d\theta}{\tan^{n-2}\theta} = \frac{-1}{(n-1)\tan^{n-1}\theta} - \int\frac{d\theta}{\tan^{n-2}\theta}. \qquad (11)$$

EXAMPLES.

1. $\int \tan^4\theta\, d\theta.$ *Ans.* $\frac{\tan^3\theta}{3} - \tan\theta + \theta.$

2. $\int \frac{d\theta}{\tan^3\theta}.$ $-\frac{1}{2\tan^2\theta} - \log(\sin\theta).$

3. $\int \frac{d\theta}{\tan^5\theta}.$ $\frac{-1}{4\tan^4\theta} + \frac{1}{2\tan^2\theta} + \log(\sin\theta).$

56. **Trigonometrical Transformations.**—Many elementary integrations are immediately reducible to one or other of the preceding formulæ of reduction, by aid of the transformations given in Art. 23. For example, if we assume $x = a\tan\theta$, the expression $\frac{x^m dx}{(a^2+x^2)^{\frac{n}{2}}}$ transforms into $\sin^m\theta \cos^{n-m-2}\theta\, d\theta$ (neglecting a constant multiplier).

In like manner, the substitution of $a\sin\theta$ for x transforms the expression $\frac{x^m\, dx}{(a^2-x^2)^{\frac{n}{2}}}$ into $\frac{a^{m-n+1}\sin^m\theta\, d\theta}{\cos^{n-1}\theta}$: and, if $x = a\sec\theta$, the expression $\frac{x^m dx}{(x^2-a^2)^{\frac{n}{2}}}$ transforms into $\frac{\cos^{n-m-2}\theta\, d\theta}{\sin^{n-1}\theta}$ (neglecting the constant multiplier).

A similar transformation may be applied in other cases; for example to find the integral of $\dfrac{x^n\,dx}{(2ax - x^2)^{\frac{1}{2}}}$.

Let $x = 2a\sin^2\theta$, then $dx = 4a\sin\theta\cos\theta\,d\theta$,

and the transformed integral is

$$2^{n+1}a^n\int\sin^{2n}\theta\,d\theta;$$

accordingly the formula of reduction is the same as that in (5).

EXAMPLES.

1. $\displaystyle\int\frac{x^4\,dx}{(1 - x^2)^{\frac{1}{2}}}$. *Ans.* $\dfrac{1\cdot 3}{2\cdot 4}\sin^{-1}x - \dfrac{x\sqrt{1-x^2}}{8}(3 + 2x^2)$.

2. $\displaystyle\int\frac{dx}{x^3\sqrt{1 - x^2}}$. $\dfrac{1}{4}\log\dfrac{1 - x}{1 + x} - \dfrac{\sqrt{1 - x^2}}{2x^2}$.

3. $\displaystyle\int\frac{dx}{(a^2 + x^2)^{\frac{5}{2}}}$. $\dfrac{x}{a^4(a^2 + x^2)^{\frac{1}{2}}} - \dfrac{x^3}{3a^4(a^2 + x^2)^{\frac{3}{2}}}$.

4. $\displaystyle\int\frac{x^4\,dx}{(a^2 + x^2)^2}$. $\dfrac{-x^3}{2(a^2 + x^2)} + \dfrac{3}{2}\left(x - a\tan^{-1}\dfrac{x}{a}\right)$.

5. $\displaystyle\int\frac{x^2\,dx}{(2ax - x^2)^{\frac{1}{2}}}$. $-(2ax - x^2)^{\frac{1}{2}}\left(\dfrac{x}{2} + \dfrac{3a}{2}\right) + 3a^2\sin^{-1}\sqrt{\dfrac{x}{2a}}$.

The integrals considered in this Article admit also of a more direct treatment. We shall commence with the following:—

57. **Cases in which $\dfrac{x^m\,dx}{(a + cx^2)^{\frac{n}{2}}}$ is immediately integrable.**

We have seen, in Art. 44, that the proposed expression is integrable immediately when m is an *odd positive integer.*

Again, when m is an even integer, if we assume $a + cx^2 = x^2 z$, the transformed expression is

$$\frac{-(z^2 - c)^{\frac{n-m-3}{2}}\,dz}{a^{\frac{n-m-1}{2}}\,z^{\frac{n-1}{2}}}.$$

This is immediately integrable when $n - m - 3$ is even and positive, i. e. when m is either an *even negative integer, or an even positive integer less than* $n - 1$.

For example $\dfrac{dx}{(a + cx^2)^{\frac{n}{2}}}$ becomes $-\dfrac{(z^2 - c)^{\frac{n-3}{2}}\,dz}{a^{\frac{n-1}{2}}\,z^{n-1}}$, and, accordingly, is always integrable by this transformation, since n is an odd integer, by hypothesis.

EXAMPLES.

1. $\displaystyle\int \frac{dx}{(a + cx^2)^{\frac{5}{2}}}.$ *Ans.* $\displaystyle\frac{x}{a^2 (a + cx^2)^{\frac{1}{2}}} \left\{1 - \frac{cx^2}{3(a + cx^2)}\right\}.$

2. $\displaystyle\int \frac{x^2\,dx}{(1 + x^2)^{\frac{7}{2}}}.$ $\displaystyle\frac{x^3}{a^2 (a + cx^2)^{\frac{3}{2}}} \left\{\frac{1}{3} - \frac{cx^2}{5(a + cx^2)}\right\}.$

3. $\displaystyle\int \frac{x^3\,dx}{(a^2 + x^2)^{\frac{5}{2}}}.$ $\displaystyle\frac{-(2a^2 + 3x^2)}{3(a^2 + x^2)^{\frac{3}{2}}}.$

4. $\displaystyle\int \frac{dx}{x^4 (a + cx^2)^{\frac{7}{2}}}.$

The differentials considered in this Article are cases of a more general class, called binomial differentials.

58. **Binomial Differentials.**—Expressions of the form

$$x^m (a + bx^n)^p\,dx,$$

in which m, n, p denote any numbers, positive, negative, or fractional, are called binomial differentials.

Such expressions can be immediately integrated in two cases, which we proceed to determine by transformations analogous of the preceding Article.

(1). Let $a + bx^n = z$; then $x = \left(\frac{z-a}{b}\right)^{\frac{1}{n}}$

and

$$dx = \frac{1}{nb}\left(\frac{z-a}{b}\right)^{\frac{1}{n}-1} dz;$$

hence

$$x^m (a + bx^n)^p\, dx = \frac{(z-a)^{\frac{m+1}{n}-1} z^p\, dz}{n\, b^{\frac{m+1}{n}}}.$$

Consequently whenever $\frac{m+1}{n}$ is a *positive integer*, the transformed expression is immediately integrable after expansion by the Binomial Theorem.

(2). Again, if we substitute $\frac{1}{y}$ for x, the differential becomes

$$y^{-np-m-2} (ay^n + b)^p\, dy.$$

This is immediately integrable as in the preceding case whenever $\frac{-(np+m+1)}{n}$ is a positive integer; i.e. when $\frac{m+1}{n} + p$ is a *negative integer*. In this latter case the integration is effected by the substitution of z for $ax^{-n} + b$.

EXAMPLES.

1. $\int \frac{x^5\, dx}{(1 + x^3)^{\frac{1}{2}}}$. *Ans.* $\frac{2(1 + x^3)^{\frac{1}{2}}(x^3 - 2)}{9}$.

2. $\int \frac{dx}{(1 + x^3)^{\frac{4}{3}}}$. $\frac{x}{(1 + x^3)^{\frac{1}{3}}}$.

3. $\int \frac{dx}{x^2 (1 + x^4)^{\frac{3}{4}}}$. $-\frac{(1 + x^4)^{\frac{1}{4}}}{x}$.

4. $\int \frac{dx}{x^{\frac{1}{2}} (1 + x^2)^{\frac{5}{4}}}$. $\frac{2x^{\frac{1}{2}}}{(1 + x^2)^{\frac{1}{4}}}$.

When neither of the preceding processes is applicable, the expression, if p be a fractional index, is, in general, incapable of integration in a finite number of terms. Before proceeding with this investigation we shall discuss a few simple forms of integration by reduction, involving transcendental functions.

59. **Reduction of** $\int e^{mx} x^n dx$,
where n is an integer.

Integrating by parts we have

$$\int x^n e^{mx} dx = \frac{x^n e^{mx}}{m} - \frac{n}{m}\int x^{n-1} e^{mx} dx. \qquad (12)$$

By successive applications of this formula the integral reduces to depend on $\int e^{mx} dx$, i. e. on $\frac{e^{mx}}{m}$.

Again, to find $\int \frac{e^{mx}}{x^n} dx$.

Assuming $u = e^{mx}$, $v = \frac{-1}{(n-1)x^{n-1}}$, and integrating by parts, we have.

$$\int \frac{e^{mx} dx}{x^n} = \frac{-e^{mx}}{(n-1)x^{n-1}} + \frac{m}{n-1}\int \frac{e^{mx} dx}{x^{n-1}}. \qquad (13)$$

By means of this the integral is reduced to depend on

$$\int \frac{e^{mx} dx}{x}.$$

The value of this integral cannot be obtained in a finite form: it however may be exhibited in the shape of an infinite series; for, expanding e^{ax} and integrating each term separately, we have

$$\int \frac{e^{mx} dx}{x} = \log x + \frac{mx}{1} + \frac{m^2 x^2}{1 \cdot 2^2} + \frac{m^3 x^3}{1 \cdot 2 \cdot 3^2} + \&c. \qquad (14)$$

The integral of $a^x x^n\,dx$ is immediately reducible to the preceding, since $a^x = e^{x \log a}$. Consequently, by the substitution of $\log a$ for m in (12) and (13), we obtain the formulæ of reduction for

$$\int a^x x^n\,dx \text{ and } \int \frac{a^x}{x^n}\,dx.$$

In like manner we have immediately

$$\int e^{-x} x^n\,dx = -e^{-x} x^n + n \int e^{-x} x^{n-1}\,dx. \qquad (15)$$

60. **Reduction of** $\int x^m (\log x)^n dx$.

Let $y = \log x$, and the integral reduces to that discussed in the last Article.

The formula of reduction is

$$\int x^m (\log x)^n\,dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1}\,dx. \qquad (16)$$

EXAMPLES.

1. $\int x^3 e^{ax}\,dx$. *Ans.* $\frac{e^{ax}}{a}\left\{x^3 - \frac{3}{a}x^2 + \frac{3 \cdot 2}{a^2}x - \frac{3 \cdot 2 \cdot 1}{a^3}\right\}$.

2. $\int x^3 (\log x)^2\,dx$. $\frac{x^4}{4}\left\{(\log x)^2 - \frac{\log x}{2} + \frac{1}{8}\right\}$.

3. $\int \frac{e^x\,dx}{x^4}$. $-\frac{e^x}{3}\left\{\frac{1}{x^3} + \frac{1}{2} + \frac{1}{2x}\right\} + \frac{1}{3 . 2 . 1}\int \frac{e^x\,dx}{x}$.

61. **Reduction of** $\int x^n \cos ax\,dx$,

$$\int x^n \cos ax\,dx = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} \sin ax\,dx.$$

again

$$\int x^{n-1} \sin ax\,dx = -\frac{x^{n-1}\cos ax}{a} + \frac{n-1}{a}\int x^{n-2} \cos ax\,dx,$$

hence

$$\int x^n \cos ax\,dx = \frac{x^{n-1}(a \sin ax + n \cos ax)}{a^2} - \frac{n(n-1)}{a^2}\int x^{n-2} \cos ax\,dx.$$

The formula of reduction for $x^n \sin ax\,dx$ can be obtained in like manner.

Again, if we substitute y for $\sin^{-1}x$, the integral

$$\int (\sin^{-1} x)^n\,dx$$

transforms into

$$\int y^n \cos y\,dy,$$

and accordingly can be integrated by the preceding formula.

EXAMPLES.

1. $\int x^3 \cos x\,dx$. *Ans.* $x^3 \sin x + 3x^2 \cos x - 3.2.x \sin x - 3.2.1.\cos x$.

2. $\int x^4 \sin x\,dx$.

Ans. $-x^4 \cos x + 4x^3 \sin x + 4.3.x^2 \cos x - 4.3.2.x \sin x - 4.3.2.1.\cos x$.

62. **Reduction of** $\int e^{ax} \cos^n x\,dx$.

Integrating by parts, we get

$$\int e^{ax} \cos^n x\,dx = \frac{\cos^n x\, e^{ax}}{a} + \frac{n}{a}\int e^{ax} \cos^{n-1} x \sin x\,dx;$$

again

$$\int e^{ax} \cos^{n-1} x \sin x\,dx = \frac{e^{ax} \cos^{n-1} x \sin x}{a}$$

$$- \frac{1}{a}\int e^{ax}\left\{\cos^n x - (n-1)\cos^{n-2} x \sin^2 x\right\}dx = \frac{e^{ax}\cos^{n-1}x \sin x}{a} +$$

$$\frac{(n-1)}{a}\int e^{ax} \cos^{n-2} x\,dx - \frac{n}{a}\int e^{ax}\cos^n x\,dx;$$

substituting and solving for $\int e^{ax} \cos^n x\,dx$, we get

$$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1}x(a\cos x + n\sin x)}{a^2+n^2} +$$

$$\frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2} x\,dx. \qquad (17)$$

The form of reduction for $e^{ax}\sin^n x\,dx$ can be obtained in like manner.

63. **Reduction of** $\int \cos^m x \sin nx\,dx$.

Integrating by parts, we get

$$\int \cos^m x \sin nx\,dx = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \cos nx \sin x\,dx;$$

replacing $\cos nx \sin x$ by $\sin nx \cos x - \sin(n-1)x$, after one or two simple transformations we get

$$\int \cos^m x \sin nx\,dx = -\frac{\cos^m x \cos nx}{m+n}$$

$$+\frac{m}{m+n}\int \cos^{m-1} x \sin(n-1)x\,dx. \qquad (18)$$

The formulæ of reduction for $\cos^m x \cos nx\,dx$, and $\sin^m x \sin nx\,dx$ can be easily found in like manner.

EXAMPLES.

1. $\int e^{ax}\sin^2 x\,dx.$ *Ans.* $\frac{e^{ax}\sin x}{4+a^2}(a\sin x - 2\cos x) + \frac{2e^{ax}}{a(4+a^2)}.$

2. $\int \cos^2 x \sin 4x\,dx.$ $-\frac{\cos^2 x \cos 4x}{6} - \frac{\cos x \,.\, \cos 3x}{12} - \frac{\cos 2x}{24}.$

3. $\int e^{-x}\cos^2 x\,dx.$ $-\frac{e^{-x}\cos x(2\sin x - \cos x)}{5} - \frac{2e^{-x}}{3}.$

64. We shall now return to the discussion of the integrals already considered in Arts. 56 & 57; and commence with the reduction of the expression $\frac{x^m\,dx}{(a+cx^2)^{\frac{1}{2}}}$. This, as well as other formulæ of reduction of the same type, is best investigated by aid of a previous differentiation.

Thus, we have

$$\frac{d}{dx}\left\{x^{m-1}(a+cx^2)^{\frac{1}{2}}\right\} = (m-1)x^{m-2}(a+cx^2)^{\frac{1}{2}} + \frac{cx^m}{(a+cx^2)^{\frac{1}{2}}}$$

$$= \frac{(m-1)\,x^{m-2}(a+cx^2) + cx^m}{(a+cx^2)^{\frac{1}{2}}}$$

$$= \frac{(m-1)\,ax^{m-2}}{(a+cx^2)^{\frac{1}{2}}} + \frac{mc\,x^m}{(a+cx^2)^{\frac{1}{2}}}.$$

hence, transposing and integrating, we obtain

$$\int \frac{x^m\,dx}{(a+cx^2)^{\frac{1}{2}}} = \frac{x^{m-1}(a+cx^2)^{\frac{1}{2}}}{mc} - \frac{(m-1)\,a}{mc}\int \frac{x^{m-2}\,dx}{(a+cx^2)^{\frac{1}{2}}}. \qquad (19)$$

By this formula the integral is reduced to one of lower dimensions; and by repetition of the same process the expression can be always integrated when m is a positive integer.

The formula (19) evidently holds whether m be positive or negative; accordingly, if we change m into $-(m-2)$, we obtain, after transposing and dividing,

$$\int \frac{dx}{x^m(a+cx^2)^{\frac{1}{2}}} = \frac{(a+cx^2)^{\frac{1}{2}}}{(m-1)ax^{m-1}} - \frac{(m-2)\,c}{(m-1)a}\int \frac{dx}{x^{m-2}(a+cx^2)^{\frac{1}{2}}}. \qquad (20)$$

65. More generally, we have

$$\frac{d}{dx}\{x^{m-1}(a+cx^2)^n\} = (m-1)\,x^{m-2}(a-cx^2)^n + 2ncx^m(a+cx^2)^{n-1}$$

$$= (a+cx^2)^{n-1}\{(m-1)a\,x^{m-2} + (m+2n-1)\,cx^m\}$$

hence

$$\int x^m(a+cx^2)^{n-1}\,dx = \frac{x^{m-1}(a+cx^2)^n}{(m+2n-1)c}$$

$$-\frac{(m-1)\,a}{(m+2n-1)\,c}\int x^{m-2}(a+cx^2)^{n-1}\,dx. \qquad (21)$$

Hence, when m is positive the integral can be reduced to one lower by two degrees. If m be negative, the formula can be transformed as in the preceding Article, and the integration reduced two degrees.

We next proceed to consider the case where n is negative.

66. **Reduction of** $\displaystyle\int \frac{x^m\,dx}{(a+cx^2)^n}$.

m and n being both positive.

Here $$\int \frac{x^m\,dx}{(a+cx^2)^n} = \int x^{m-1}\frac{x\,dx}{(a+cx^2)^n}.$$

Let $x^{m-1} = u$, and $\displaystyle\int \frac{x\,dx}{(a+cx^2)^n} = v$,

or $$\frac{-1}{2(n-1)c(a+cx^2)^{n-1}} = v,$$

and we get

$$\int \frac{x^m\,dx}{(a+cx^2)^n} = \frac{-x^{m-1}}{2(n-1)c(a+cx^2)^{n-1}} + \frac{m-1}{2(n-1)c}\int \frac{x^{m-2}\,dx}{(a+cx^2)^{n-1}}. \quad (22)$$

By successive applications of this form the integral admits of being reduced to another of a simpler shape. As observed already, the complete integral cannot be determined in general, unless when n is either an integer, or is of the form $\frac{r}{2}$ where r is an integer.

67. **Reduction of** $\displaystyle\int \frac{x^m\,dx}{(a+2bx+cx^2)^{\frac{1}{2}}}$.

By differentiation, we have

$$\frac{d}{dx}\{x^{m-1}(a+2bx+cx^2)^{\frac{1}{2}}\} = (m-1)x^{m-2}(a+2bx+cx^2)^{\frac{1}{2}}$$

$$+ \frac{x^{m-1}(b+cx)}{(a+2bx+cx^2)^{\frac{1}{2}}} = \frac{(m-1)ax^{m-2}+(2m-1)bx^{m-1}+mcx^m}{(a+2bx+cx^2)^{\frac{1}{2}}};$$

hence $$\int\frac{x^m\,dx}{(a+2bx+cx^2)^{\frac{1}{2}}}=\frac{x^{m-1}(a+2bx+cx^2)^{\frac{1}{2}}}{mc}$$
$$-\frac{(2m-1)\,b}{mc}\int\frac{x^{m-1}dx}{(a+2bx+cx^2)^{\frac{1}{2}}}-\frac{(m-1)\,a}{mc}\int\frac{x^{m-2}\,dx}{(a+2bx+cx^2)^{\frac{1}{2}}}. \quad (23)$$

This furnishes the formula of reduction for this case: by successive applications, the integral depends ultimately on those of

$$\frac{x\,dx}{(a+2bx+cx^2)^{\frac{1}{2}}}\text{ and }\frac{dx}{(a+2bx+cx^2)^{\frac{1}{2}}}.$$

These have been determined already in Arts. 9 and 12.

Again, the integral of $\dfrac{dx}{x^m(a+2bx+cx^2)^{\frac{1}{2}}}$ can be reduced to the preceding form by making $x=\dfrac{1}{z}$.

68. The more general integral

$$\frac{x^m\,dx}{(a+2bx+cx^2)^n}.$$

admits of being treated in like manner.

For if $a+2bx+cx^2$ be represented by T, we have, by differentiation,

$$\frac{d}{dx}\left(\frac{x^{m-1}}{T^{n-1}}\right)=\frac{(m-1)\,x^{m-2}}{T^{n-1}}-\frac{2(n-1)x^{m-1}(b+cx)}{T^n}$$
$$=\frac{(m-1)\,x^{m-2}(a+2bx+cx^2)-2\,(n-1)\,x^{m-1}(b+cx)}{T^n}$$
$$=\frac{(m-1)\,ax^{m-2}}{T^n}+\frac{2b\,(m-n)\,x^{m-1}}{T^n}-\frac{(2n-m-1)\,cx^m}{T^n}.$$

Hence, we get the formula of reduction

$$\int\frac{x^m dx}{T^n}=\frac{-x^{m-1}}{(2n-m-1)\,c\;T^{n-1}}+\frac{2\,(m-n)b}{(2n-m-1)c}\int\frac{x^{m-1}\,dx}{T^n}$$
$$+\frac{(m-1)a}{(2n-m-1)\,c}\int\frac{x^{m-2}dx}{T^n}. \quad (24)$$

By aid of this, the integral of $\frac{x^m\,dx}{T^n}$, when m is a positive integer, is made to depend on those of $\frac{x\,dx}{T^n}$ and $\frac{dx}{T^n}$. Again, it is easily seen that the integral of $\frac{x\,dx}{T^n}$ is reduced to that of $\frac{dx}{T^n}$: for

$$\int\frac{x\,dx}{T^n} = \frac{1}{c}\int\frac{(b+cx)\,dx}{T^n} - \frac{b}{c}\int\frac{dx}{T^n}$$

$$= \frac{-1}{(2(n-1)c\,T^{n-1}} - \frac{b}{c}\int\frac{dx}{T^n}. \qquad (25)$$

69. In order to reduce $\int\frac{dx}{T^n}$, we have

$$\frac{d}{dx}\left(\frac{b+cx}{T^n}\right) = \frac{c}{T^n} - \frac{2n(b+cx)^2}{T^{n+1}}$$

$$= \frac{c}{T^n} + \frac{2n(ac-b^2)}{T^{n+1}} - \frac{2nc}{T^n} = \frac{2n(ac-b^2)}{T^{n+1}} - \frac{(2n-1)\,c}{T^n}.$$

Hence $$\int\frac{dx}{T^{n+1}} = \frac{b+cx}{2n(ac-b^2)T^n} + \frac{(2n-1)\,c}{2n(ac-b^2)}\int\frac{dx}{T^n}. \qquad (26)$$

By aid of this formula of reduction the integral of $\frac{dx}{T^n}$ can be found, whenever n is an integer, or when it is of the form $\frac{r}{2}$, (r being an integer).

70 **Reduction of** $\int\frac{dx}{(a+b\cos x)^n}$, when n is a positive integer.

Let $U = a + b\cos x$, then $\frac{dU}{dx} = -b\sin x$, $\cos x = \frac{U-a}{b}$.

Then, by differentiation, we have

$$\frac{d}{dx}\left\{\frac{\sin x}{U^{n-1}}\right\} = \frac{\cos x}{U^{n-1}} + \frac{(n-1)\,b\sin^2 x}{U^n}$$

$$= \frac{\cos x}{U^{n-1}} + \frac{(n-1)\,b}{U^n} - \frac{(n-1)\,b\cos^2 x}{U^n},$$

substitute $\dfrac{U-a}{b}$ for $\cos x$ in the numerators of these fractions, and we get

$$\frac{d}{dx}\left\{\frac{\sin x}{U^{n-1}}\right\} = \frac{1}{b\,U^{n-2}} - \frac{a}{b\,U^{n-1}} + \frac{(n-1)b}{U^n} - \frac{n-1}{b\,U^{n-2}} + \frac{2(n-1)\,a}{b\,U^{n-1}}$$

$$- \frac{(n-1)a^2}{bU^n} = \frac{-(n-2)}{bU^{n-2}} + \frac{(2n-3)a}{bU^{n-1}} - \frac{(n-1)\,(a^2-b^2)}{bU^n}.$$

Hence, transposing and integrating, we get

$$\int\frac{dx}{U^n} = \frac{-b\sin x}{(n-1)\,(a^2-b^2)\,U^{n-1}} + \frac{(2n-3)\,a}{(n-1)\,(a^2-b^2)}\int\frac{dx}{U^{n-1}}$$

$$\frac{n-2}{(n-1)\,(a^2-b^2)}\int\frac{dx}{U^{n-2}}. \qquad (27)$$

By this formula the proposed integral can be reduced to depend on

$$\int\frac{dx}{a+b\cos x},$$

the value of which has been found in Art. 18.

71. The integral considered in the last Article can also be found by aid of a transformation, as follows:

$$\frac{dx}{(a+b\cos x)^n} = \frac{dx}{\left\{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}\right\}^n}$$

$$= \frac{dx}{\left(A\cos^2\frac{x}{2} + B\sin^2\frac{x}{2}\right)^n}$$

$$= \frac{\left(1 + \tan^2 \frac{x}{2}\right)^n dx}{\left(A + B \tan^2 \frac{x}{2}\right)^n},$$

(where $A = a + b$, $B = a - b$).

Next, assume $\tan \frac{x}{2} = \sqrt{\frac{A}{B}} \tan \phi$, then

$\left(1 + \tan^2 \frac{x}{2}\right) dx = 2\sqrt{\frac{A}{B}} (1 + \tan^2 \phi)\, d\phi$: and we get

$$\frac{\left(1 + \tan^2 \frac{x}{2}\right)^n dx}{\left(A + B \tan^2 \frac{x}{2}\right)^n} = 2\sqrt{\frac{A}{B}} \cdot \frac{\left(1 + \frac{A}{B} \tan^2 \phi\right)^{n-1} d\phi}{A^n \sec^{2n-2} \phi}$$

$$= 2\, \frac{(B \cos^2 \phi + A \sin^2 \phi)^{n-1} d\phi}{(AB)^{n-\frac{1}{2}}}.$$

Hence, replacing A and B by $a + b$ and $a - b$, we get

$$\int \frac{dx}{(a + b \cos x)^n} = 2 \int \frac{(a - b \cos 2\phi)^{n-1} d\phi}{(a^2 - b^2)^{n-\frac{1}{2}}}. \qquad (28)$$

When n is a positive integer, the integral at the right-hand side can be found by expanding $(a - b \cos 2\phi)^{n-1}$, and integrating each term separately by formula (4).

72. **Integration of** $\frac{f(x)}{\phi(x)} \frac{dx}{\sqrt{a + 2bx + cx^2}}$.

We shall conclude this chapter with the discussion of the above form, where $f(x)$ and $\phi(x)$ are supposed rational algebraic functions of x.

If $f(x)$ be of higher dimensions than $\phi(x)$, the fraction may be written in the form

$$\frac{f(x)}{\phi(x)} = Q + \frac{R}{\phi(x)}.$$

Again, since Q is of the form $p + qx + rx^2 + \&c.$ the integration of $\dfrac{Qdx}{\sqrt{a+2bx+cx^2}}$ can be found by the method of Art. 67.

The fraction $\dfrac{R}{\phi(x)}$ can be decomposed by the method of partial fractions (Chap. II.). To any root, α, which is not a multiple root, corresponds a term of the form $\dfrac{A}{x-\alpha}$: and the corresponding term in the expression under discussion is

$$\frac{A\,dx}{(x-\alpha)\sqrt{a+2bx+cx^2}}:$$

the method of integration of this has been given in Art. 13.

Next. To a multiple root correspond terms of the form

$$\frac{Bdx}{(x-\alpha)^r\sqrt{a+2bx+cx^2}}.$$

This is reducible to the form of Art. 67, on making $x - \alpha = \dfrac{1}{z}$. Again, to a pair of imaginary roots corresponds an expression of the form

$$\frac{(lx+m)\,dx}{\{(x-\alpha)^2+\beta^2\}\sqrt{a+2bx+cx^2}}.$$

If z be substituted for $x - \alpha$, the transformed expression may be written

$$\frac{(Lz+M)\,dz}{(z^2+\beta^2)\sqrt{A+2Bz+Cz^2}},$$

where L, M, A, B, C, are constants.

To integrate this form; assume* $z = \beta \tan(\theta + \gamma)$, where θ is a new variable, and γ an arbitrary constant, and the transformed expression is

$$\frac{\{L\beta \sin(\theta+\gamma) + M\cos(\theta+\gamma)\}\, d\theta}{\beta\sqrt{A\cos^2(\theta+\gamma) + 2B\beta\cos(\theta+\gamma)\sin(\theta+\gamma) + C\beta^2\sin^2(\theta+\gamma}}.$$

Again, the expression under the square root is easily transformed into

$$\tfrac{1}{2}\{A + C\beta^2 + (A - C\beta^2)\cos 2(\theta+\gamma) + 2B\beta\sin 2(\theta+\gamma)\}$$

$$= \tfrac{1}{2}\Big[A + C\beta^2 + \cos 2\theta\,\{(A - C\beta^2)\cos 2\gamma + 2B\beta\sin 2\gamma\}$$

$$+ \sin 2\theta\,\{2B\beta\cos 2\gamma - (A - C\beta^2)\sin 2\gamma\}\Big].$$

Moreover, since γ is perfectly arbitrary, it may be assumed so as to satisfy the equation

$$2B\beta\cos 2\gamma - (A - C\beta^2)\sin 2\gamma = 0, \text{ or } \tan 2\gamma = \frac{2B\beta}{A - C\beta^2}:$$

and consequently the proposed expression is reducible to the form

$$\frac{(L'\cos\theta + M'\sin\theta)\, d\theta}{\sqrt{P + Q\cos 2\theta}},$$

(in which L', M', P and Q are constants), or

$$\frac{L'd(\sin\theta)}{\sqrt{P + Q - 2Q\sin^2\theta}} - \frac{M'd(\cos\theta)}{\sqrt{P - Q + 2Q\cos^2\theta}},$$

each of which is immediately integrable.

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* For this simple method of determining the integral in question I am indebted to Mr. Cathcart.

EXAMPLES.

1. $\int \cos^3\theta \sin 2\theta \, d\theta.$ *Ans.* $\frac{2}{5}\cos^5\theta.$

2. $\int \sin^2\theta \cos^3\theta \, d\theta.$ $\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5}.$

3. $\int \sin^5\theta \cos^5\theta \, d\theta.$ $-\frac{1}{64}\left\{\cos 2\theta - \frac{2}{3}\cos^3 2\theta + \frac{1}{5}\cos^5 2\theta\right\}.$

4. $\int \frac{\cos^4\theta \, d\theta}{\sin\theta}.$ $\frac{\cos^3\theta}{3} + \cos\theta + \log\left(\tan\frac{\theta}{2}\right).$

5. $\int \frac{\cos^4\theta \, d\theta}{\sin^3\theta}.$ $\left(\cos^3\theta - \frac{3}{2}\cos\theta\right)\frac{1}{\sin^2\theta} - \frac{3}{2}\log\tan\left(\frac{\theta}{2}\right)$

6. $\int \frac{dx}{(1+x^2)^{\frac{7}{2}}}.$ $\left(\frac{4.2}{5.3}x^4 + \frac{4}{3}x^2 + 1\right)\frac{x}{(1+x^2)^{\frac{5}{2}}}.$

7. $\int x^{2n-1}(a+bx^n)^p \, dx.$ $\frac{(a+bx^n)^{p+1}\{(p+1)bx^n - a\}}{n(p+1)(p+2)b^2}.$

8. $\int e^{-x}\cos^3 x \, dx.$ $\frac{e^{-x}}{10}\left\{3(\sin x - \cos x) + \cos^2 x(3\sin x - \cos x)\right\}$

9. If $\int \frac{d\theta}{\sin^m\theta \cos^n\theta} = \frac{A}{\sin^{m-1}\theta \cos^{n-1}\theta} + B\int \frac{d\theta}{\sin^{m-2}\theta \cos^n\theta},$
determine the values of A and B by differentiation.

10. $\int \frac{(x^2 - a^2)\,dx}{(x^2 + a^2)^3}.$

11. $\int \frac{\sin^2\theta \, d\theta}{(1+\cos\theta)^2}.$ *Ans.* $2\tan^2\frac{\theta}{2} - \theta.$

12. Prove that the integral $\int \frac{\sin^m\theta \, d\theta}{(1+\cos\theta)^n}$ transforms into $2^{m-n+1}\int \frac{\sin^m\phi \, d\phi}{\cos^{2n-m}\phi},$ where $\theta = 2\phi.$

13. $\int \frac{dx}{(a + b\cos x)^2}$

$$Ans.\ \frac{-b\sin x}{(a^2 - b^2)(a + b\cos x)} + \frac{2a}{(a^2 - b^2)^{\frac{3}{2}}} \tan^{-1}\left\{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} \tan\frac{x}{2}\right\}.$$

14. $\int \frac{\cos\theta\, d\theta}{(5 + 4\cos\theta)^2}$. Ans. $\frac{5}{9}\frac{\sin\theta}{5 + 4\cos\theta} - \frac{8}{27}\tan^{-1}\left(\frac{\tan\frac{\theta}{2}}{3}\right)$.

15 $$\int (\sin^{-1} x)^4\, dx = x\,\{(\sin^{-1} x)^4 - 4.3.(\sin^{-1} x)^2 + 4.3.2.1\}$$

$$+ 4\sqrt{1 - x^2}\sin^{-1} x\,\{(\sin^{-1} x)^2 - 3.2\}.$$

16. Prove by Art. 70, that any expression of the form $\frac{f(\cos x)\,dx}{(a + b\cos x)^n}$ is capable of being integrated when $f(\cos x)$ consists of integral powers of $\cos x$.

17. Show, in like manner, that the expression

$$\frac{f(\cos x, \sin x)\,dx}{(a + b\cos x)^n},$$

can be integrated when $f(\cos x, \sin x)$ consists only of integral powers of $\cos x$ and $\sin x$.

18. If $$\int \frac{(A + Bx + Cx^2)\,dx}{(a + \beta x)(a + bx + cx^2)} = P\log(a + \beta x) + Q\log(a + bx + cx^2)$$

$$+ R\int \frac{dx}{a + bx + cx^2},$$

find the values of P, Q, and R.

19. $\int \frac{d\theta}{(a\cos^2\theta + b\sin^2\theta)^2}$. Ans. $\frac{(a + b)\,\phi}{(ab)^{\frac{3}{2}}} + \frac{(a - b)\sin 2\phi}{2\,(ab)^{\frac{3}{2}}}$.

where $\tan\phi = \sqrt{\frac{b}{a}}\tan\theta$.

20. The formula of reduction in (24) fails when $m = 2n - 1$; show how to perform the integration in this case.

CHAPTER IV.

INTEGRATION BY RATIONALIZATION.

73. **Integration of Monomials.**—If an algebraic expression contain fractional powers of the variable x, it can evidently be rendered rational, by assuming $x = z^n$; where n is the least common multiple of the denominators of the several fractional powers. By this means the integration of such expressions is reduced to that of rational functions.

For example, to find

$$\int \frac{(1 + x^{\frac{1}{4}})\,dx}{1 + x^{\frac{1}{2}}}.$$

Let $x = z^4$, and the transformed expression is

$$4\int \frac{z^3 (1 + z)\,dz}{1 + z^2}.$$

Consequently the value of the integral is

$$\frac{4x^{\frac{3}{4}}}{3} + 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4\tan^{-1}(x^{\frac{1}{4}}) - 2\log(1 + x^{\frac{1}{2}}).$$

Again, any algebraic expression containing integral powers of x along with irrational powers of an expression of the form $a + bx$, is immediately reduced to the preceding, by the substitution of z for $a + bx$.

Examples.

1. $\displaystyle\int \frac{x^3 dx}{\sqrt{x - 1}}$. *Ans.* $\displaystyle\frac{2\sqrt{x-1}}{5 \cdot 7}[5x^3 + 6x^2 + 8x + 16]$.

2. $\displaystyle\int \frac{x dx}{(a + bx)^{\frac{3}{2}}}$. " $\displaystyle\frac{2}{b^2}\frac{(2a + bx)}{\sqrt{a + bx}}$.

3. $\displaystyle\int \frac{dx}{x + \sqrt{x - 1}}$. " $\displaystyle\log(x + \sqrt{x-1}) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt{x-1}+1}{\sqrt{3}}\right)$

74. **Rationalization of** $F(x, \sqrt{a + 2bx + cx^2})\,dx$. It has been observed (Art. 28), that the integration, in a finite form, of irrational expressions containing powers of x beyond the second, is in general impossible, without introducing new transcendental functions. We shall accordingly restrict our investigation to the case of an algebraic function containing a single radical of the form $\sqrt{a + 2bx + cx^2}$, where a, b, c, are any constants, positive or negative.

Integrals of this form have been already treated by the method of Reduction (Art. 72). We shall discuss them here by the method of rationalization.

The expression[*] $\dfrac{f(x)}{\phi(x)} \dfrac{dx}{\sqrt{a + 2bx + cx^2}}$ can be made rational in several ways; which we propose to consider in order.

(1). Assume $\sqrt{a + 2bx + cx^2} = z - x\sqrt{c}$. (1)

Then $a + 2bx = z^2 - 2xz\sqrt{c}$, $\therefore\ b\,dx = z\,dz - \sqrt{c}\,(x\,dz + z\,dx)$,

or $dx\,(b + z\sqrt{c}) = dz\,(z - x\sqrt{c}) = dz\sqrt{a + 2bx + cx^2}$.

$$\therefore\ \frac{dx}{\sqrt{a + 2bx + cx^2}} = \frac{dz}{b + z\sqrt{c}}. \tag{2}$$

Also
$$x = \frac{z^2 - a}{2(b + z\sqrt{c})}. \tag{3}$$

This substitution obviously renders the proposed expression rational; and its integration is reducible to the class considered in Chapter II.

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[*] It will be shown subsequently that the integration of all expressions of the form

$$F(x, \sqrt{a + 2bx + cx^2})\,dx$$

is reducible to that of the above: when F is a rational algebraic function.

It may also be observed that the most expeditious method of integration in practice is that of successive Reduction (Arts. 67, 68).

When $b = 0$, we get

$$\frac{dx}{\sqrt{a + cx^2}} = \frac{dz}{z\sqrt{c}}, \text{ and } x = \frac{z^2 - a}{2z\sqrt{c}} \text{ (see Art. 9).}$$

By aid of the preceding substitution the expression

$$\frac{dx}{(x + p)\sqrt{a + 2bx + cx^2}} \text{ (Art. 13),}$$

transforms into
$$\frac{dz}{z^2 + 2zp\sqrt{c} - a + 2pb}.$$

For example, to find $\int \frac{dx}{(a + \beta x)\sqrt{1 + x^2}}$.

Here, $x = \frac{z^2 - 1}{2z}$, and $\frac{dx}{(a + \beta x)\sqrt{1 + x^2}} = \frac{2dz}{\beta z^2 + 2az - \beta}$.

$$\therefore \int \frac{dx}{(a + \beta x)\sqrt{1 + x^2}} = \frac{1}{\sqrt{a^2 + \beta^2}} \log\left(\frac{\beta z + a - \sqrt{a^2 + \beta^2}}{\beta z + a + \sqrt{a^2 + \beta^2}}\right)$$

When the coefficient c is negative, the preceding method introduces imaginaries; we proceed to other transformations to which this objection does not apply.

(2). Assume[*] $\sqrt{a + 2bx + cx^2} = \sqrt{a} + xz.$ (4)

Squaring both sides, we get immediately

$$2b + cx = 2z\sqrt{a} + xz^2,$$

$$\therefore\ dx(c - z^2) = 2dz(\sqrt{a} + xz) = 2dz\sqrt{a + 2bx + cx^2}.$$

Hence
$$\frac{dx}{\sqrt{a + 2bx + cx^2}} = \frac{2dz}{c - z^2}. \qquad (5)$$

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[*] This is reducible to the preceding, by changing x into $\frac{1}{y}$, and then employing the former transformation.

And
$$x = \frac{2(z\sqrt{a} - b)}{c - z^2}. \tag{6}$$

This substitution also evidently renders the proposed expression rational, provided a be positive.

For example, to find

$$\int \frac{dx}{x\sqrt{1 - x^2}}.$$

Assume $\sqrt{1 - x^2} = 1 + xz$, and we get

$$\int \frac{dx}{x\sqrt{1 - x^2}} = \int \frac{dz}{z} = \log z = \log\left(\frac{1 - \sqrt{1 - x^2}}{x}\right).$$

(3). Again, when the roots of $a + 2bx + cx^2$ are real, there is another method of transformation.

For, let α and β be the roots, and the radical becomes either of the form

$$\sqrt{c(x - \alpha)(x - \beta)}, \text{ or } \sqrt{c(x - \alpha)(\beta - x)},$$

according as the coefficient of x^2 is positive or negative.

In the former case, assume $\sqrt{x - \alpha} = z\sqrt{x - \beta}$, and we get

$$x = \frac{\alpha - \beta z^2}{1 - z^2}; \text{ hence } x - \beta = \frac{\alpha - \beta}{1 - z^2}, \therefore \frac{dx}{x - \beta} = \frac{2z\,dz}{1 - z^2}.$$

Accordingly

$$\frac{dx}{\sqrt{c(x - \alpha)(x - \beta)}} = \frac{dx}{z(x - \beta)\sqrt{c}} = \frac{2}{\sqrt{c}} \frac{dz}{1 - z^2}. \tag{7}$$

In the latter case, let $\sqrt{x - \alpha} = z\sqrt{\beta - x}$, and we get

$$x = \frac{\alpha + \beta z^2}{1 + z^2},$$

and
$$\frac{dx}{\sqrt{c(x - \alpha)(\beta - x)}} = \frac{2}{\sqrt{c}} \frac{dz}{1 + z^2}. \tag{8}$$

For example, the integral

$$\int \frac{dx}{(a + \beta x)\sqrt{1 - x^2}}$$

transforms into

$$\int \frac{2dz}{(a + \beta) z^2 + a - \beta},$$

on making $x = \dfrac{z^2 - 1}{z^2 + 1}$.

The student can compare this method of integrating the preceding example with that of Art. 13, and he will find no difficulty in identifying the results.

It may be observed that in the application of the foregoing methods, it is advisable that the student should in each case select whatever method avoids the introduction of imaginaries.

Thus, as already observed, the first should only be employed when c is positive: in like manner, the second requires a to be positive: and the third, that the roots be real.

It is easily seen that when a and c are both negative, the roots must be real; for, the expression

$$\sqrt{- a + 2bx - cx^2} \text{ or } \sqrt{b^2 - ac - (cx - b)^2}$$

is imaginary for all real values of x, unless $b^2 - ac$ is positive: i.e. unless the roots are real.

Accordingly the third method is always applicable when the other two fail.

From the preceding investigation it follows that the expression

$$F(x, \sqrt{a + 2bx + cx^2})\, dx$$

can be always rationalized; F denoting a rational algebraic function of x and of $\sqrt{a + 2bx + cx^2}$.

EXAMPLES.

1. $\displaystyle\int \frac{dx}{(2 + 3x)\sqrt{4 - x^2}}$. *Ans.* $\displaystyle\frac{1}{2\sqrt{2}} \log \frac{\sqrt{4 + 2x} - \sqrt{2 - x}}{\sqrt{4 + 2x} + \sqrt{2 - x}}$.

2. $\displaystyle\int \frac{dx}{\overline{(a^2 + x^2)^{\frac{1}{2}} + x)}^{\frac{1}{2}}}$.

Assume $z = (a^2 + x^2)^{\frac{1}{2}} + x$, and we get for the value of the proposed integral

$$\frac{2}{3} z^{\frac{3}{2}} - \frac{2}{5} \frac{a^2}{z^{\frac{5}{2}}}.$$

3. $\int dx \sqrt{x + \sqrt{2 + x^2}}$. *Ans.* $\frac{2}{3} \frac{x^2 + x\sqrt{2 + x^2} - 2}{\sqrt{x + \sqrt{2 + x^2}}}$.

4. $\int x^m \{(a^2 + x^2)^{\frac{1}{2}} + x\}^n dx$.

Making the same assumption as in Ex. 2, the transformed expression is

$$\frac{(z^2 - a^2)^m \, (a^2 + z^2)\, dz}{2^{m+1} \, z^{m-n+2}},$$

which is immediately integrable when m is a *positive* integer.

5. $\int \frac{dx}{\{(1 + x^2)^{\frac{1}{2}} - x\}^n}$. $\frac{[(1 + x^2)^{\frac{1}{2}} + x)]^{n+1}}{2(n + 1)} + \frac{[(1 + x^2)^{\frac{1}{2}} + x]^{n-1}}{2(n - 1)}$.

6. $\int \frac{\{(1 + x^2)^{\frac{1}{2}} + x)\}^n dx}{(1 - x^2)^{\frac{1}{2}}}$ $\frac{1}{n} \{(1 + x^2)^{\frac{1}{2}} + x\}^n$.

7. $\int \frac{dx}{ax + \beta\sqrt{a + cx^2}}$.

75. **General Investigation.**—The following more general investigation of the preceding may be worthy of the notice of the student.

Let R denote the quadratic expression $a + 2bx + cx^2$; then, since the even powers of $\sqrt{R}$ are rational, and the odd contain $\sqrt{R}$ as a factor, any rational algebraic function of x and of $\sqrt{R}$ can evidently be reduced to the form

$$\frac{P + Q\sqrt{R}}{P' + Q'\sqrt{R}},$$

where P, Q, P', Q' are rational algebraic functions of x.

On multiplying the numerator and denominator of this fraction by the complementary surd $P' - Q'\sqrt{R}$, the denominator becomes rational, and the resulting expression may be written in the form

$$M + N\sqrt{R},$$

where M and N are rational functions

The integration of Mdx is effected by the method of Chapter II.

Also
$$\int N\sqrt{R}\,dx = \int \frac{NR\,dx}{\sqrt{R}};$$

which is of the form

$$\int \frac{f(x)\,dx}{\phi(x)\sqrt{a + 2bx + cx^2}}.$$

Let as before $\sqrt{a + 2bx + cx^2} = \sqrt{c\,(x - \alpha)\,(x - \beta)}$, and substitute $\dfrac{\lambda + 2\mu z + \nu z^2}{\lambda' + 2\mu' + \nu' z^2}$ instead of x, when the radical becomes

$$\frac{\sqrt{c\,\{\lambda - \alpha\lambda' + 2(\mu - \alpha\mu')z + (\nu - \alpha\nu')z^2\}\{\lambda - \beta\lambda' + 2(\mu - \beta\mu')z + (\nu - \beta\nu')z^2\}}}{\lambda' + 2\mu' z + \nu' z^2}. \tag{9}$$

Again, if the quadratic factors under this radical be made each a perfect square, the expression obviously becomes rational.

The simplest method of fulfilling these conditions is, by reducing one factor to a constant, and the other to the term containing z^2.

Accordingly, let

$$\lambda - \alpha\lambda' = 0, \quad \mu - \alpha\mu' = 0, \quad \mu - \beta\mu' = 0, \quad \nu - \beta\nu' = 0;$$

or
$$\mu = 0, \quad \mu' = 0, \quad \lambda = \alpha\lambda', \quad \nu = \beta\nu'.$$

On making these substitutions the expression (9) becomes

$$\frac{(\beta - \alpha)\,z\sqrt{-c\lambda'\nu'}}{\lambda' + \nu' z^2}, \text{ while } x = \frac{\alpha\lambda' + \beta\nu' z^2}{\lambda' + \nu' z^2}.$$

In order that $\sqrt{-c\lambda'\nu'}$ should be real, λ' and ν' must have opposite signs, when c is positive; and the same sign when c is negative.

It is also easily seen that without loss[*] of generality we may assume $\lambda' = 1$, and $\nu' = \pm 1$.

Hence, when c is positive, we get $x = \dfrac{a - \beta z^2}{1 - z^2}$, and when c is negative, $x = \dfrac{a + \beta z^2}{1 + z^2}$.

These agree with the third transformation in the preceding Article.

More generally, when the factors in (9) are each squares, we must have

$$(\mu - a\mu')^2 - (\lambda - a\lambda')(\nu - a\nu') = 0,$$

or
$$\mu^2 - \lambda\nu + (\lambda\nu' + \nu\lambda' - 2\mu\mu')\,a + (\mu'^2 - \lambda'\nu')\,a^2 = 0. \qquad (10)$$

and a similar equation with β instead of a.

Moreover, by hypothesis, a satisfies the equation

$$a + 2ba + ca^2 = 0.$$

Accordingly (10) is satisfied if we assume the constants λ, μ, &c., so as to satisfy the equations

$$\mu^2 - \lambda\nu = a, \quad \lambda'\nu + \lambda\nu' - 2\mu\mu' = 2b, \quad \mu'^2 - \lambda'\nu' = c. \qquad (11)$$

Again, solving for z from the equation

$$x(\lambda' + 2\mu' z + \nu' z^2) = \lambda + 2\mu z + \nu z^2, \qquad (12)$$

we obtain

$$(\nu - x\nu')z + \mu - x\mu' = \sqrt{\mu^2 - \lambda\nu + (\lambda\nu' + \lambda'\nu - 2\mu\mu')x + (\mu'^2 - \lambda'\nu')x^2}$$
$$= \sqrt{a + 2bx + cx^2}. \qquad (13)$$

---

[*] For the substitution of y^2 for $\dfrac{\nu' z^2}{\lambda'}$ transforms

$$\frac{a\lambda' + \beta\nu' z^2}{\lambda' + \nu' z^2} \text{ into } \frac{a + \beta y^2}{1 + y^2}, \therefore \text{ \&c.}$$

Also, by differentiation, we get from (12),

$$(\lambda' + 2\mu' z + \nu' z^2)\, dx = 2\,\{\mu + \nu z - x(\mu' + \nu' z)\}\, dz$$

$$= 2\sqrt{a + 2bx + cx^2}\, dz.$$

$$\therefore \quad \frac{dx}{\sqrt{a + 2bx + cx^2}} = \frac{2dz}{\lambda' + 2\mu' z + \nu' z^2}. \qquad (14)$$

Now, since we have but three equations (11) connecting λ, μ, &c.; they can be satisfied in an indefinite number of ways.

We proceed to consider the simplest cases for real transformations.

(1). Let a be positive, and we may assume $\nu = 0$, and $\mu' = 0$; this gives

$$\mu = \sqrt{a}, \quad \lambda\nu' = 2b, \quad \lambda'\nu' = -c.$$

Again, without loss of generality, we may assume $\nu' = -1$, which gives

$$\lambda = -2b, \quad \lambda' = c; \text{ whence } x = \frac{2(z\sqrt{a} - b)}{c - z^2},$$

and

$$\frac{dx}{\sqrt{a + 2bx + cx^2}} = \frac{2dz}{c - z^2}.$$

These agree with the results in (5) and (6).

(2). In like manner if c be positive, we may assume

$$\nu' = 0, \quad \mu = 0, \quad \text{and } \nu' = 1,$$

which gives

$$\mu' = \sqrt{c}, \quad \lambda = -a, \quad \text{and } \lambda' = 2b.$$

$$\therefore \; x = \frac{z^2 - a}{2(b + z\sqrt{c})} \text{ and } \frac{dx}{\sqrt{a + 2bx + cx^2}} = \frac{dz}{b + z\sqrt{c}};$$

as in (2) and (3).

It may be observed that since these results do not contain the roots α and β, they hold whether these roots be real or imaginary; as already shown in Art. 74.

It is easily seen that if we make $\mu = 0$ and $\mu' = 0$, we get the third transformation.

76. If the expression to be integrated be of the form

$$\frac{f(x)dx}{\sqrt{a + 2bx + cx^2}},$$

where $f(x)$ is a *rational algebraic* function of x, it is often more convenient to proceed as follows:—

The substitution of $z - \frac{b}{c}$ for x transforms the proposed into

$$\frac{f\left(z - \frac{b}{c}\right) dz}{\sqrt{a' + cz^2}},$$

where $$a' = \frac{ac - b^2}{c}.$$

If the even and odd powers be separated in the expansion of $f\left(z - \frac{b}{c}\right)$, it can plainly be written in the form

$$\phi(z^2) + z\psi(z^2),$$

and the proposed integral becomes

$$\int \frac{\phi(z^2)\,dz}{\sqrt{a' + cz^2}} + \int \frac{z\psi(z^2)\,dz}{\sqrt{a' + cz^2}}.$$

The former of these is rationalized (Art. 25), by making $\sqrt{a' + cz^2} = yz$, and the latter by making $\sqrt{a' + cz^2} = y$.

It may be observed that in general the expression

$$\frac{f(x^2)}{\phi(x^2)} \frac{dx}{\sqrt{a + cx^2}},$$

is also evidently made rational by the transformation

$$\sqrt{a + cx^2} = xy.$$

77. **Case of a Recurring Biquadratic under the Radical Sign.**—As the solution of a recurring equation of the fourth degree is immediately reducible to that of a quadratic, it is natural to consider in what cases an Elliptic Integral (Art. 28) in which the biquadratic under the radical sign is recurring, is reducible by the corresponding substitution.

Writing the expression in the form

$$\frac{\phi(x)\,dx}{\sqrt{a+2bx+cx^2+2bx^3+ax^4}}, \text{ or } \frac{\phi(x)\,dx}{x\sqrt{a\left(x^2+\frac{1}{x^2}\right)+2b\left(x+\frac{1}{x}\right)+c}},$$

and, assuming $x+\frac{1}{x}=z$, the radical becomes $\sqrt{az^2+2bz+c-2a}$:

and also $$\frac{dx}{x}\left(x-\frac{1}{x}\right)=dz.$$

Consequently, in order that the transformed equation should be of the required type, it is obvious that $\phi(x)$ must be reducible to the form

$$f\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right).$$

In this case $$\frac{f\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)dx}{\sqrt{a+2bx+cx^2+2bx^3+ax^4}}$$

transforms into $$\frac{f(z)\,dz}{\sqrt{az^2+2bz+c-2a}}.$$

In like manner, the expression

$$\frac{f\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)dx}{\sqrt{a+2bx+cx^2-2bx^3+ax^4}}$$

transforms into $\dfrac{f(z)\,dz}{\sqrt{az^2-2bz+2a-c}}$, by the assumption $x-\frac{1}{x}=z$.

When $b=0$, the expression can in some cases be reduced by assuming

$$x^2+\frac{1}{x^2}=z, \text{ or } x^2-\frac{1}{x^2}=z.$$

EXAMPLES.

1. $\int \frac{(x^2-1)\,dx}{x\sqrt{1+x^4}}$. *Ans.* $\log \frac{1+x^2+\sqrt{1+x^4}}{x}$.

2. $\int \frac{(x^2+1)\,dx}{x\sqrt{1+x^4}}$. ,, $\log \frac{x^2-1+\sqrt{1+x^4}}{x}$.

3. $\int \frac{1-x^2}{1+x^2}\,\frac{dx}{\sqrt{1+x^4}}$. ,, $\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x\sqrt{2}}{1+x^2}\right)$.

4. $\int \frac{1+x^2}{1-x^2}\,\frac{dx}{\sqrt{1+x^4}}$. ,, $\frac{1}{\sqrt{2}} \log \frac{\sqrt{1+x^4}+\sqrt{2}}{1-x^2}$

This and the preceding were given by Euler (Cal. Int. Tom. 4): the connexion, however, of their solution with the method of recurring equations does not appear to have been pointed out by him.

5. $\int \frac{(x^4-1)\,dx}{x^2\sqrt{x^4+x^2+1}}$. *Ans.* $\frac{\sqrt{x^4+x^2-1}}{x}$.

Let $x^2+\frac{1}{x^2}=z$.

6. $\int \frac{(x^2-1)\,dx}{x\sqrt{(x^2+ax+1)\,(x^2+\beta x+1)}}$.

Ans. $2\log \frac{\sqrt{x^2+ax+1}+\sqrt{x^2+\beta x+1}}{x}$.

7. $\int \frac{(1-x^2)\,dx}{(1+x^2)\sqrt{1+x^2+x^4}}$. *Ans.* $\sin^{-1}\left(\frac{x}{1+x^2}\right)$.

EXAMPLES.

1. $\displaystyle\int \frac{x\,dx}{(a+bx)^{\frac{1}{3}}}$. *Ans.* $\dfrac{3(2bx-3a)}{10b^2}(a+bx)^{\frac{2}{3}}$.

2. $\displaystyle\int \frac{1+x^2}{1-x^2}\,\frac{dx}{\sqrt{1+x^2+x^4}}$. ,, $\dfrac{1}{\sqrt{3}}\log\dfrac{\sqrt{1+x^2+x^4}+x\sqrt{3}}{x^2-1}$

3. $\displaystyle\int \frac{dx}{(1+x^4)\{(1+x^4)^{\frac{1}{2}}-x^2\}^{\frac{1}{2}}}$. ,, $\sin^{-1}\left(\dfrac{x}{(1+x^4)^{\frac{1}{4}}}\right)$.

Assume $x = (1+x^4)^{\frac{1}{4}}\sin\theta$, &c.

4. $\displaystyle\int \frac{dx}{(1+x^{2n})\{(1+x^{2n})^{\frac{1}{n}}-x^2\}^{\frac{1}{2}}}$. ,, $\sin^{-1}\left(\dfrac{x}{(1+x^{2n})^{\frac{1}{n}}}\right)$.

5. $\displaystyle\int \frac{x\,dx}{(1+x)^{\frac{1}{2}}+(1+x)^{\frac{1}{3}}}$.

Assume $1+x = z^6$.

6. $\displaystyle\int \frac{x^2\,dx}{(1-x^4)(1+x^4)^{\frac{1}{2}}}$.

Ans. $\dfrac{1}{4\sqrt{2}}\log\dfrac{(1+x^4)^{\frac{1}{2}}+x\sqrt{2}}{1-x^2}+\dfrac{1}{4\sqrt{2}}\tan^{-1}\dfrac{(1+x^4)^{\frac{1}{2}}}{x\sqrt{2}}$.

7. $\displaystyle\int \frac{(1+x^4)\,dx}{(1-x^4)^{\frac{3}{2}}}$. *Ans.* $\dfrac{x}{(1-x^4)^{\frac{3}{2}}}$.

8. $\displaystyle\int \frac{(1+x^4)^{\frac{1}{2}}\,dx}{1-x}$. $\dfrac{1}{2\sqrt{2}}\log\left(\dfrac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}\right)+\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{\sqrt{1+x^4}}$

9. $\displaystyle\int \frac{1-x^2}{1+ax+x}\,\frac{dx}{\sqrt{1+ax+bx^2+ax^3+x^4}}$.

10. $\displaystyle\int \frac{1-a^2x^2}{1+a^2x^2}\,\frac{dx}{\sqrt{1+2cx^2+a^2x^4}}$.

Ans. $\dfrac{1}{\sqrt{2(c-a)}}\log\dfrac{x\sqrt{2(c-a)}+\sqrt{1+2cx^2+a^2x^4}}{1+a^2x^2}$, when $c > a$.

,, $\dfrac{1}{\sqrt{2(a-c)}}\sin^{-1}\left(\dfrac{\sqrt{2(a-c)}}{1+a^2x^2}\right)$, when $a > c$.

EXAMPLES.

1. $\displaystyle\int \frac{(x^2 - 1)\,dx}{x\sqrt{1 + x^4}}.$ *Ans.* $\displaystyle\log \frac{1 + x^2 + \sqrt{1 + x^4}}{x}.$

2. $\displaystyle\int \frac{(x^2 + 1)\,dx}{x\sqrt{1 + x^4}}.$ „ $\displaystyle\log \frac{x^2 - 1 + \sqrt{1 + x^4}}{x}.$

3. $\displaystyle\int \frac{1 - x^2}{1 + x^2}\,\frac{dx}{\sqrt{1 + x^4}}.$ „ $\displaystyle\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x\sqrt{2}}{1 + x^2}\right).$

4. $\displaystyle\int \frac{1 + x^2}{1 - x^2}\,\frac{dx}{\sqrt{1 + x^4}}.$ „ $\displaystyle\frac{1}{\sqrt{2}} \log \frac{\sqrt{1 + x^4} + \sqrt{2}}{1 - x^2}$

This and the preceding were given by Euler (Cal. Int. Tom. 4): the connexion, however, of their solution with the method of recurring equations does not appear to have been pointed out by him.

5. $\displaystyle\int \frac{(x^4 - 1)\,dx}{x^2\sqrt{x^4 + x^2 + 1}}.$ *Ans.* $\displaystyle\frac{\sqrt{x^4 + x^2 - 1}}{x}.$

Let $\displaystyle x^2 + \frac{1}{x^2} = z.$

6. $\displaystyle\int \frac{(x^2 - 1)\,dx}{x\sqrt{(x^2 + ax + 1)(x^2 + \beta x + 1)}}.$

Ans. $\displaystyle 2\log \frac{\sqrt{x^2 + ax + 1} + \sqrt{x^2 + \beta x + 1}}{x}.$

7. $\displaystyle\int \frac{(1 - x^2)\,dx}{(1 + x^2)\sqrt{1 + x^2 + x^4}}.$ *Ans.* $\displaystyle\sin^{-1}\left(\frac{x}{1 + x^2}\right).$

EXAMPLES.

1. $\displaystyle\int \frac{x\,dx}{(a+bx)^{\frac{1}{3}}}$. *Ans.* $\displaystyle\frac{3(2bx-3a)}{10b^2}(a+bx)^{\frac{2}{3}}$.

2. $\displaystyle\int \frac{1+x^2}{1-x^2}\,\frac{dx}{\sqrt{1+x^2+x^4}}$. ,, $\displaystyle\frac{1}{\sqrt{3}}\log\frac{\sqrt{1+x^2+x^4}+x\sqrt{3}}{x^2-1}$

3. $\displaystyle\int \frac{dx}{(1+x^4)\{(1+x^4)^{\frac{1}{2}}-x^2\}^{\frac{1}{2}}}$. ,, $\displaystyle\sin^{-1}\left(\frac{x}{(1+x^4)^{\frac{1}{4}}}\right)$.

Assume $x = (1+x^4)^{\frac{1}{4}}\sin\theta$, &c.

4. $\displaystyle\int \frac{dx}{(1+x^{2n})\{(1+x^{2n})^{\frac{1}{n}}-x^2\}^{\frac{1}{2}}}$. ,, $\displaystyle\sin^{-1}\left(\frac{x}{(1+x^{2n})^{\frac{1}{n}}}\right)$.

5. $\displaystyle\int \frac{x\,dx}{(1+x)^{\frac{1}{2}}+(1+x)^{\frac{1}{3}}}$.

Assume $1+x=z^6$.

6. $\displaystyle\int \frac{x^2\,dx}{(1-x^4)(1+x^4)^{\frac{1}{2}}}$.

Ans. $\displaystyle\frac{1}{4\sqrt{2}}\log\frac{(1+x^4)^{\frac{1}{2}}+x\sqrt{2}}{1-x^2}+\frac{1}{4\sqrt{2}}\tan^{-1}\frac{(1+x^4)^{\frac{1}{2}}}{x\sqrt{2}}$.

7. $\displaystyle\int \frac{(1+x^4)\,dx}{(1-x^4)^{\frac{3}{2}}}$. *Ans.* $\displaystyle\frac{x}{(1-x^4)^{\frac{3}{2}}}$.

8. $\displaystyle\int \frac{(1+x^4)^{\frac{1}{2}}\,dx}{1-x}$. $\displaystyle\frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}\right)+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{1+x^4}}$

9. $\displaystyle\int \frac{1-x^2}{1+ax+x}\,\frac{dx}{\sqrt{1+ax+bx^2+ax^3+x^4}}$.

10. $\displaystyle\int \frac{1-a^2x^2}{1+a^2x^2}\,\frac{dx}{\sqrt{1+2cx^2+a^2x^4}}$.

Ans. $\displaystyle\frac{1}{\sqrt{2(c-a)}}\log\frac{x\sqrt{2(c-a)}+\sqrt{1+2cx^2+a^2x^4}}{1+a^2x^2}$, when $c > a$.

,, $\displaystyle\frac{1}{\sqrt{2(a-c)}}\sin^{-1}\left(\frac{\sqrt{2(a-c)}}{1+a^2x^2}\right)$, when $a > c$.

CHAPTER V.

MISCELLANEOUS EXAMPLES OF INTEGRATION.

78. **Integration of** $\dfrac{(A\cos x + B\sin x + C)\,dx}{a\cos x + b\sin x + c}$.

Let $a\cos x + b\sin x + c = u$, then $-a\sin x + b\cos x = \dfrac{du}{dx}$. Now assume

$$A\cos x + B\sin x + C = \lambda u + \mu\frac{du}{dx} + \nu,$$

and equating coefficients we have

$$A = \lambda a + \mu b, \quad B = \lambda b - \mu a, \quad C = \lambda c + \nu.$$

Solving for λ, μ, ν, we get

$$\lambda = \frac{Aa + Bb}{a^2 + b^2}, \quad \mu = \frac{Ab - Ba}{a^2 + b^2}, \quad \nu = C - \frac{(Aa + Bb)\,c}{a^2 + b^2}.$$

Hence $$\int\frac{(A\cos x + B\sin x + C)\,dx}{a\cos x + b\sin x + c} = \frac{(Aa + Bb)\,x}{a^2 + b^2}$$

$$+ \frac{Ab - Ba}{a^2 + b^2}\log(a\cos x + b\sin x + c)$$

$$+ \frac{(a^2 + b^2)\,C - (Aa + Bb)c}{a^2 + b^2}\int\frac{dx}{a\cos x + b\sin x + c}.$$

The latter integral can be readily found; for, if we make $a = r\cos\alpha$, $b = r\sin\alpha$, we get

$$a\cos x + b\sin x = r(\cos x\cos\alpha + \sin x\sin\alpha) = r\cos(x - \alpha).$$

On making $x - a = \theta$, the integral reduces to the form considered in Art. 18.

As a simple example, let us take

$$\int \frac{(A + B \tan x)\, dx}{a + b \tan x}.$$

Here
$$\frac{A + B \tan x}{a + b \tan x} = \frac{A \cos x + B \sin x}{a \cos x + b \sin x},$$

and we evidently have

$$\int \frac{(A + B \tan x)\, dx}{a + b \tan x} = \frac{(Aa + Bb)\, x}{a^2 + b^2} + \frac{Ab - Ba}{a^2 + b^2} \log (a \cos x + b \sin x).$$

79. **Integration of** $\dfrac{f(\cos x, \sin x)\, dx}{a \cos x + b \sin x + c}$;

where f is a rational algebraic function, not involving fractions.

As in the preceding Article, assume $x = \theta + a$, and the expression becomes of the form

$$\frac{\phi\,(\cos \theta, \sin \theta)\, d\theta}{A \cos \theta + B}.$$

Again, since $\sin^2\theta = 1 - \cos^2\theta$, any integral function of $\sin \theta$ and $\cos \theta$ can be transformed into another of the shape

$$\phi_1 (\cos \theta) + \sin \theta\, \phi_2 (\cos \theta).$$

Accordingly, the proposed expression is reducible to

$$\frac{\phi_1 (\cos \theta)\, d\theta}{A \cos \theta + B} + \frac{\phi_2 (\cos \theta) \sin \theta\, d\theta}{A \cos \theta + B}.$$

The latter is immediately integrable, by assuming

$$A \cos \theta + B = z.$$

To integrate the former, we divide by $A \cos \theta + B$, and integrate each term separately.

80. **Integration of**

$$\frac{f(\cos x)\,dx}{(a_1 + b_1 \cos x)\,(a_2 + b_2 \cos x)\;\ldots\;(a_n + b_n \cos x)}$$

where f, as before, denotes a rational algebraic function.

Substitute z for $\cos x$, and decompose

$$\frac{f(z)}{(a_1 + b_1 z)\,(a_2 + b_2 z)\;\ldots\;(a_n + b_n z)}$$

by the method of partial fractions: then the expression to be integrated reduces to the sum of a number of terms of the form

$$\frac{dx}{A + B \cos x},$$

each of which can be immediately integrated.

EXAMPLES.

1. $\displaystyle\int \frac{dx}{\cos x\,(5 + 3\cos x)}$. *Ans.* $\displaystyle\frac{1}{10}\log\left(\frac{1 + \sin x}{1 - \sin x}\right) - \frac{3}{10}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right)$.

2. $\displaystyle\int \frac{dx}{\sin^2 x\,(a + b\cos x)}$, when $a > b$.

Ans. $\displaystyle\frac{b - a\cos x}{(a^2 - b^2)\sin x} - \frac{b^2}{(a^2 - b^2)^{\frac{3}{2}}}\cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right)$.

3. $\displaystyle\int \frac{dx}{\cos^2 x\,(a + b\cos x)}$. " $\displaystyle\frac{\tan x}{a} + \frac{b}{a^2}\log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + \frac{b^2}{a^2}\int\frac{dx}{a + b\cos x}$.

81. **Integration of** $\{f(x) + f'(x)\}\,e^x\,dx$.

The expression $e^x P dx$ is immediately integrable whenever P can be divided into the sum of two functions, one of which is the derived of the other.

For, let $P = f(x) + f'(x)$,

then $\int e^x P dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx.$

Again, integrating by parts,

$$\int e^x f(x)\, dx = f(x)\, e^x - \int e^x f'(x)\, dx.$$

Accordingly

$$\int \{f(x) + f'(x)\}\, e^x dx = e^x f(x).$$

For instance, to find

$$\int e^x \frac{x}{(1+x)^2}\, dx.$$

Here
$$\frac{x}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2},$$

and consequently the value of the proposed is $\dfrac{e^x}{1+x}$.

EXAMPLES.

1. $\int e^x (\cos x + \sin x)\, dx.$ *Ans.* $e^x \sin x.$

2. $\int e^x \frac{1 + x \log x}{x}\, dx.$,, $e^x \log x.$

3. $\int e^x \frac{x^2+1}{(x+1)^2}\, dx.$,, $e^x \frac{x-1}{x+1}.$

4. $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx.$,, $\frac{e^x}{1+x^2}.$

82. **Differentiation under the Sign of Integration.**—The integral of any expression of the form $\phi(x, a)\, dx$, where a is independent of x, is obviously a function of a as well as of x. Suppose the integral to be denoted by $F(x, a)$, i. e. let

$$F(x, a) = \int \phi(x, a)\, dx,$$

then
$$\frac{d}{dx}\{F(x, a)\} = \phi(x, a).$$

Again, differentiating both sides with respect to a, we have, since x and a are independent,

$$\frac{d^2 . F(x, a)}{da\, dx} = \frac{d . \phi(x, a)}{da},$$

or (Art. 117, Dif. Cal.),

$$\frac{d}{dx}\left(\frac{d . F(x, a)}{da}\right) = \frac{d . \phi(x, a)}{da}.$$

Consequently, integrating with respect to x, we get

$$\frac{d . F(x, a)}{da} = \int \frac{d . \phi(x, a)}{da}\, dx,$$

$$\text{i. e. } \frac{d}{da} \int \phi(x, a)\, dx = \int \frac{d\phi(x, a)}{da}\, dx. \qquad (1)$$

In other words, if

$$u = \int \phi(x, a)\, dx,$$

then
$$\frac{du}{da} = \int \frac{d\phi}{da}\, dx,$$

provided a be independent of x, in which case, accordingly, it is permitted to *differentiate under the sign of integration.*

By continuing the same process of reasoning we obviously get

$$\frac{d^n u}{da^n} = \int \frac{d^n \phi(x, a)}{da^n}\, dx, \qquad (2)$$

where $u = \int \phi(x, a)\, dx$, a being independent of x.

For example, if the equation

$$\int e^{ax}\, dx = \frac{e^{ax}}{a},$$

differentiated n times with respect to a, we get

$$\int x^n e^{ax}\,dx = \left(\frac{d}{da}\right)^n \left(\frac{e^{ax}}{a}\right).$$

$$= e^{ax}\left(x + \frac{d}{da}\right)^n \left(\frac{1}{a}\right).$$

See Arts. 48, 59, Dif. Cal.).

Again, in Art. 21 we have seen that

$$\int e^{ax} \sin mx\,dx = \frac{e^{ax}(a \sin mx - m \cos mx)}{m^2 + a^2}.$$

Accordingly

$$\int x^n e^{ax} \sin mx\,dx = \left(\frac{d}{da}\right)^n . \left(\frac{e^{ax}(a \sin mx - m \cos mx)}{m^2 + a^2}\right).$$

We now proceed to consider the inverse process, the aplication of integration under the sign of integration.

83. **Integration under the Sign of Integration.**—If in the st Article we suppose $\phi(x, a)$ to be the derived with respect a of another function v, i. e.

$$\phi(x, a) = \frac{dv}{da},$$

hen

$$v = \int \phi(x, a)\,da.$$

Also by the preceding Article we have

$$\frac{d}{da}\left(\int v dx\right) = \int \frac{dv}{da}\,dx = \int \phi(x, a)\,dx = F(x, a).$$

Hence $\int v dx = \int F(x, a)\,da.$

In other words, if

$$F(x, a) = \int \phi(x, a)\,dx,$$

then
$$\int F(x, a)\,da = \int \left[\int \phi(x, a)\,da\right] dx. \qquad (3)$$

It may be remarked that the results established in this and the preceding Article are chiefly of importance in connexion with definite integrals. Some examples of such application will be given in the next chapter.

84. **Integration by Infinite Series.**—It has been already observed that in most cases we fail in exhibiting the integral of any proposed expression in finite terms. In such cases we can often represent the integral in the form of an infinite series.

An example of an integral exhibited in such a form has been given in Art. 59.

The simplest mode of seeking the integral of $f(x)\,dx$ in the form of an infinite series, consists in expanding $f(x)$ in a series of ascending powers of x, and integrating each term separately: then if the series be convergent, it represents the integral proposed.

It can be easily seen that if the expansion of $f(x)$ be a convergent series, that of $\int f(x)\,dx$ is also convergent.

For let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n + \&c.,$$

then

$$\int f(x)\,dx = a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \ldots + \frac{a_n x^{n+1}}{n+1} + \ldots$$

Now (Dif. Cal. Art. 71), the expansion for $f(x)$ is convergent, provided that $\frac{a_n x}{a_{n-1}}$ is less than unity, for all values of n beyond a certain number; and the latter series is convergent provided $\frac{n}{n+1}\frac{a_n x}{a_{n-1}}$ be less than unity, under the same conditions.

Accordingly, the latter series is convergent whenever the former is so.

EXAMPLES.

1. $$\int \frac{dx}{\sqrt{1-x^5}} = \frac{x}{1} + \frac{1}{2}\frac{x^6}{6} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^{11}}{11} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^{16}}{16} + \&c.$$

2. $$\int \frac{dx}{\sqrt{\sin x}} = 2\sqrt{\sin x}\left(1 + \frac{1}{2}\frac{\sin^2 x}{5} + \frac{1 \cdot 3}{2 \cdot 4}\frac{\sin^4 x}{9} + \dots\right).$$

3. $$\int (1 + cx^n)^{\frac{p}{q}} x^{m-1}\, dx = x^m \left(\frac{1}{m} + \frac{pc}{q}\frac{x^n}{m+n} + \frac{p(p-q)}{1 \cdot 2 \cdot q^2}\frac{x^{2n}}{m+2n} + \&c.\right).$$

85. **Expansion of** $\int \log(1 + 2m\cos x + m^2)\, dx$.

We shall conclude by showing that the integral

$$\int \log(1 + 2m\cos x + m^2)\, dx$$

can be exhibited in the form of an infinite series.

For, we have

$$1 + 2m\cos x + m^2 = (1 + me^{x\sqrt{-1}})(1 + me^{-x\sqrt{-1}}).$$

Hence

$$\log(1 + 2m\cos x + m^2) = \log(1 + me^{x\sqrt{-1}}) + \log(1 + me^{-x\sqrt{-1}})$$

$$= m(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}) - \frac{m^2}{2}(e^{2x\sqrt{-1}} + e^{-2x\sqrt{-1}}) + \&c.$$

$$= 2\left(m\cos x - \frac{m^2}{2}\cos 2x + \frac{m^3}{3}\cos 3x + \&c.\right).$$

Accordingly

$$\log(1 + 2m\cos x + m^2)\, dx = 2\left(m\sin x - m^2\frac{\sin 2x}{2^2} + m^3\frac{\sin 3x}{3^2} - \dots\right). \tag{4}$$

This series becomes divergent when m is greater than unity: in that case the corresponding series can however be easily attained.

For $$1 + 2m\cos x + m^2 = m^2\left(1 + \frac{e^{x\sqrt{-1}}}{m}\right)\left(1 + \frac{e^{-x\sqrt{-1}}}{m}\right),$$

and accordingly

$$\log(1 + 2m\cos x + m^2) = 2\log m + 2\left(\frac{\cos x}{m} - \frac{\cos 2x}{2m^2} + \frac{\cos 3x}{3m^3} - \&c.\right).$$

Consequently, when $m > 1$, we have

$$\int \log(1+2m\cos x+m^2)\,dx = 2x\log m + 2\left(\frac{\sin x}{m} - \frac{\sin 2x}{2^2m^2} + \frac{\sin 3x}{3^2m^3} - \ldots\right).$$

From the above it is easily seen that the integral

$$\int \log(1 + a\cos x)\,dx$$

can be exhibited in the form of an infinite series, when a is less than unity: for making $a = \dfrac{2m}{1+m^2}$ we have

$$\log(1 + a\cos x) = \log(1 + 2m\cos x + m^2) - \log(1 + m^2).$$

The relation between m and a admits of being exhibited in a simple form; for let $a = \sin\alpha$, and we get $m = \tan\dfrac{\alpha}{2}$.

Making this substitution in (4) we get

$$\int \log(1 + \sin\alpha\cos x)\,dx = 2x\log\left(\cos\frac{\alpha}{2}\right)$$

$$+ 2\left(\tan\frac{\alpha}{2}\sin x - \tan^2\frac{\alpha}{2}\,\frac{\sin 2x}{2^2} + \&c.\right) \qquad (5)$$

EXAMPLES.

1. $\displaystyle\int \frac{(2\cos x + 3\sin x)\,dx}{3\cos x + 2\sin x}.$ *Ans.* $\displaystyle\frac{12x}{13} - \frac{5}{13}\log(3\cos x + 2\sin x).$

2. $\displaystyle\int \frac{d\theta}{1 - \sin^4\theta}.$ „ $\displaystyle\frac{1}{2}\tan\theta + \frac{1}{2\sqrt{2}}\tan^{-1}(\tan\theta\sqrt{2}).$

3. $\displaystyle\int \frac{e^x(x^3 + x + 1)\,dx}{(1 + x^2)^{\frac{3}{2}}}.$ „ $\displaystyle e^x\frac{x}{\sqrt{1 + x^2}}.$

4. $$\int \frac{d\theta}{\sin 2\theta - \sin\theta} = \frac{1}{6}\log(1 + \cos\theta) - \frac{1}{2}\log(1 - \cos\theta) + \frac{2}{3}\log(1 - 2\cos\theta).$$

5. $\displaystyle\int \frac{d\theta}{a\sin 2\theta + \beta\sin\theta}.$

Let $z = \cos\theta$, and the expression reduces to

$$\frac{-dz}{(1 - z^2)(\beta + 2az)}.$$

6. $$\int \frac{\sin\frac{\theta}{2}\tan\frac{\theta}{2}\,d\theta}{\cos\theta} = \log\left(\frac{1 - \sin\frac{\theta}{2}}{1 + \sin\frac{\theta}{2}}\right) + \frac{1}{2\sqrt{2}}\log\left(\frac{\sqrt{2}\sin\frac{\theta}{2} + 1}{\sqrt{2}\sin\frac{\theta}{2} - 1}\right).$$

7. When $x^2 < 1$, prove that

$$\int \frac{dx}{\sqrt{1 + x^4}} = \frac{x}{1} - \frac{1}{2}\frac{x^5}{5} + \frac{1\cdot 3}{2\cdot 4}\frac{x^9}{9} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^{13}}{13} + \ldots,$$

and when $x^2 > 1$

$$\int \frac{dx}{\sqrt{1 + x^4}} = -\frac{1}{x} + \frac{1}{2}\frac{1}{5x^5} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{9x^9} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{13x^{13}} - \ldots$$

8. Prove that

$$\int \frac{e^{ax}}{b + x}\,dx = e^{-ab}\left\{\log(b + x) + \frac{a}{1}\frac{b + x}{1} + \frac{a^2}{1\cdot 2}\frac{(b + x)^2}{2} + \ldots\right\},$$

and determine when the series is convergent, and when divergent.

9. Prove that

$$\int \frac{e^{\lambda\omega} + e^{-\lambda\omega}}{2} \sin^{\mu}\omega \, d\omega = \frac{\sin^{\mu+1}\omega}{\mu + 1} + \frac{\lambda^2 + 1^2}{1 \, . \, 2} \, . \, \frac{\sin^{\mu+3}\omega}{\mu + 3}$$

$$+ \frac{(\lambda^2 + 1^2)\,(\lambda^2 + 3^2)}{1 \, . \, 2 \, . \, 3 \, . \, 4} \, \frac{\sin^{\mu+5}\omega}{\mu + 5} + \ldots$$

Substitute x for $\sin\omega$, and differentiate the expansion of $e^{\lambda \sin^{-1} x}$ (Dif. Cal. Art. 85), &c.

10. $$\int \frac{e^{\lambda\omega} - e^{-\lambda\omega}}{2} \sin^{\mu}\omega \, d\omega = \frac{\lambda}{1} \, . \, \frac{\sin^{\mu+2}\omega}{\mu + 2} + \frac{\lambda\,(\lambda^2 + 2^2)}{1 \, . \, 2 \, . \, 3} \, . \, \frac{\sin^{\mu+4}\omega}{\mu + 4}$$

$$+ \frac{\lambda\,(\lambda^2 + 2^2)\,(\lambda^2 + 4^2)}{1 \, . \, 2 \, . \, 3 \, . \, 4 \, . \, 5} \, . \, \frac{\sin^{\mu+6}\omega}{\mu + 6} + \&\text{c.}$$

CHAPTER VI.

DEFINITE INTEGRALS.

86. **Integration regarded as Summation.**—We have in the commencement observed that the process of integration may be regarded as that of finding the limit of the sum of the series of values of a differential $f(x)\,dx$, when x varies by indefinitely small increments from any one assigned value to another.

It is in this aspect that the practical importance of integration chiefly consists. For example, in seeking the area of a curve, we conceive it divided into an indefinite number of suitable elementary areas, of which we seek to determine the sum by a process of integration. Applications of finding areas by this method will be given in the next chapter.

We first proceed to show the connexion between the process of integration regarded from this point of view, and that from which we have hitherto considered it.

Suppose $\phi(x)$ to represent a function of x which is *finite and continuous* for all values of x between the limits X and x_0; suppose also that $X - x_0$ is divided into n intervals $x_1 - x_0$, $x_2 - x_1$, $x_3 - x_2$, . . . $X - x_{n-1}$; then by definition (Dif. Cal. Art. 6), we have

$$\frac{\phi(x_1) - \phi(x_0)}{x_1 - x_0} = \phi'(x_0),$$

in the limit when $x_1 = x_0$; accordingly we have

$$\phi(x_1) - \phi(x_0) = (x_1 - x_0)\,(\phi'(x_0) + \epsilon_0),$$

where ϵ_0 becomes infinitely small along with $x_1 - x_0$. Hence, we may write

$$\begin{aligned}
\phi(x_1) - \phi(x_0) &= (x_1 - x_0)\,\{\phi'(x_0) + \epsilon_0\} \\
\phi(x_2) - \phi(x_1) &= (x_2 - x_1)\,\{\phi'(x_1) + \epsilon_1\} \\
\phi(x_3) - \phi(x_2) &= (x_3 - x_2)\,\{\phi'(x_2) + \epsilon_2\}. \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\phi(X) - \phi(x_{n-1}) &= (X - x_{n-1})\,\{\phi'(x_{n-1}) + \epsilon_{n-1}\}.
\end{aligned}$$

where $\varepsilon_0, \varepsilon_1 \ldots \varepsilon_{n-1}$ become evanescent when the intervals are taken as infinitely small.

By addition we have

$$\phi(X) - \phi(x_0) = (x_1 - x_0)\phi'(x_0) + (x_2 - x_1)\phi'(x_1) + \ldots$$
$$+ (X - x_{n-1})\phi'(x_{n-1}) + (x_1 - x_0)\varepsilon_0 + (x_2 - x_1)\varepsilon_1 + \ldots + (X - x_{n-1})\varepsilon_{n-1}.$$

Now if η denote the greatest of the quantities $\varepsilon_0, \varepsilon_1, \ldots \varepsilon_{n-1}$, the latter sum is evidently less than $(X - x_0)\eta$; and accordingly becomes evanescent ultimately; (compare Dif. Cal. Art. 39).

Hence

$$\phi(X) - \phi(x_0) = \text{limit of } [(x_1 - x_0)\phi'(x_0) + (x_2 - x_1)\phi'(x_1) + \ldots$$
$$+ (X - x_{n-1})\phi'(x_{n-1})]$$

when n is increased indefinitely.

This result can also be written in the form

$$\phi(X) - \phi(x_0) = \Sigma\phi'(x)\,dx,$$

where the sign of summation Σ is supposed to extend through all values of x between the limits x_0 and X.

87. **Definite Integrals, Limits of Integration.**—The result just arrived at is usually written in the form

$$f(X) - f(x_0) = \int_{x_0}^{X} f'(x)\,dx. \qquad (2)$$

In this X is called the *superior*, and x_0 the *inferior limit* of the integral.

Again, the expression

$$\int_{x_0}^{X} \phi(x)\,dx$$

is called the *definite integral* of $\phi(x)\,dx$ between the limits x_0 and X and, as shown above, represents the limit of the sum of the infinitely small elements $\phi(x)\,dx$, taken between the proposed limits.

From equation (1) we see that the limit of

$$(x_1 - x_0)f'(x_0) + (x_2 - x_1)f'(x_1) + \ldots + (X - x_{n-1})f'(x_{n-1}),$$

when $x_1 - x_0$, $x_2 - x_1$, . . . $X - x_{n-1}$ become evanescent, is got by finding the integral of $f'(x)\,dx$ (i. e. the function of which $f'(x)$ is the derived), and substituting the limits x_0, X for x in it, and subtracting the value for the lower limit from that for the upper.

If we write x instead of X in (2) we have

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(x)\,dx, \tag{3}$$

in which the upper limit* x may be regarded as variable. Again, as the lower limit x_0 may be assumed arbitrarily, $f(x_0)$ may have any value, and may be regarded as an arbitrary constant. This agrees with the results hitherto arrived at.

In contradistinction, the name *indefinite integrals* is often applied to integrals such as have been considered in the previous chapters, in which the form of the function is merely taken into account, without regard to any assigned limits.

As already observed, the definite integral of any expression between assigned limits is at once found, whenever the indefinite integral is known.

EXAMPLES.

1. $\int_a^b x^n\,dx$. *Ans.* $\dfrac{b^{n+1} - a^{n+1}}{n+1}$

2. $\int_0^{\frac{\pi}{4}} \dfrac{\sin\theta\,d\theta}{\cos^2\theta}$. " $\sqrt{2} - 1$.

3. $\int_0^a \dfrac{dx}{\sqrt{x+a} + \sqrt{x}}$. " $\frac{4}{3}\sqrt{a}\,(\sqrt{2} - 1)$.

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* The student should observe that in (3) the letter x which stands for the superior limit, and the x in the element $f'(x)\,dx$ must be considered as being entirely distinct. The want of attention to this distinction often causes much confusion in the mind of the beginner.

4. $\int_0^\infty \frac{dx}{a^2 + x^2}$. *Ans.* $\frac{\pi}{2a}$.

5. $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}$ „ $\frac{\pi}{2}$.

6. $\int_0^\infty e^{-ax}\,dx$, ($a$ positive). „ $\frac{1}{a}$.

7. $\int_0^1 \frac{dx}{1 + 2x\cos\phi + x^2}$. „ $\frac{\phi}{2\sin\phi}$.

8. $\int_0^\infty \frac{dx}{1 + 2x\cos\phi + x^2}$. „ $\frac{\phi}{\sin\phi}$.

9. $\int_0^\infty e^{-ax}\sin mx\,dx$. „ $\frac{m}{a^2 + m^2}$.

10. $\int_0^\infty e^{-ax}\cos mx\,dx$. „ $\frac{a}{a^2 + m^2}$.

11. $\int_{-\infty}^{+\infty} \frac{dx}{a + 2bx + cx^2} = \frac{\pi}{\sqrt{ac - b^2}}$, when $ac - b^2$ is positive.

88. **Change of Limits.**—If a and b be written for the limiting values of x, equation (2) becomes

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

It should be observed that it is not necessary that the increment dx should be regarded as positive, for we may regard x as decreasing by successive stages, as well as increasing.

Accordingly we have

$$\int_b^a f'(x)\,dx = f(a) - f(b) = -\int_a^b f'(x)\,dx.$$

That is, *the interchange of the limits is equivalent to a change of sign of the definite integral.*

Also, it is obvious that

$$\int_a^c \phi(x)\,dx = \int_b^c \phi(x)\,dx + \int_a^b \phi(x)\,dx;$$

and so on.

Again, if we assume x to be any function of a new variable z, so that $\phi(x)\,dx$ becomes $\psi(z)\,dz$, we obviously have

$$\int_{x_0}^{X} \phi(x)\,dx = \int_{z_0}^{Z} \psi(z)\,dz,$$

where Z and z_0 are the values which z assumes when X and x_0 are substituted for x, respectively.

For example, if $x = a \tan z$, the expression $\dfrac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$ becomes $\dfrac{\cos z\,dz}{a^2}$; and if the limits of x be 0 and ∞, those of z are 0 and $\dfrac{\pi}{2}$. Consequently

$$\int_0^\infty \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{1}{a^2}\int_0^{\frac{\pi}{2}} \cos z\,dz = \frac{1}{a^2}.$$

Also, if we substitute $a - z$ for x, we have

$$\int_0^a \phi(x)\,dx = -\int_a^0 \phi(a - z)\,dz = \int_0^a \phi(a - z)\,dz.$$

Since neither x nor z occur in the result, this equation may evidently be written in the form

$$\int_0^a \phi(x)\,dx = \int_0^a \phi(a - x)\,dx. \qquad (4)$$

We shall next proceed to evaluate a few elementary definite integrals.

89. **Values of** $\int_0^\pi \sin mx \sin nx\,dx$ and $\int_0^\pi \cos mx \cos nx\,dx$.

Since $2 \sin mx \sin nx = \cos(m - n)x - \cos(m + n)x$, and $$2\cos mx \cos nx = \cos(m-n)x + \cos(m+n)x,$$

we have

$$\int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)},$$

and $$\int \cos mx \cos nx dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}.$$

Hence, when m and n are unequal integers,

$$\int_0^\pi \sin mx \sin nx = 0, \text{ and } \int_0^\pi \cos mx \cos nx = 0. \tag{5}$$

When $m = n$, we have

$$\int \sin^2 nx\, dx = \int \frac{1 - \cos 2nx}{2} dx = \frac{x}{2} - \frac{\sin 2nx}{4n},$$

$$\therefore \int_0^\pi \sin^2 nx dx = \frac{\pi}{2}, \text{ when } n \text{ is an integer.}$$

In like manner we have

$$\int_0^\pi \cos^2 nx\, dx = \frac{\pi}{2}.$$

90. **To prove that**

$$\int_0^1 x^{n-1}(1-x)^{m-1} dx = \int_0^1 x^{m-1}(1-x)^{n-1} dx = \frac{1 \cdot 2 \cdot 3 \dots (m-1)}{n(n+1)\dots(n+m-1)}, \tag{6}$$

when m and n are positive, and m is an integer.

The first relation is evident from (4).

Again, integrating by parts, we have

$$\int x^{n-1}(1-x)^{m-1} dx = \frac{x^n}{n}(1-x)^{m-1} + \frac{m-1}{n}\int x^n (1-x)^{m-2} dx.$$

Moreover, since n and $m - 1$ are positive, the term $x(1-x)^{m-1}$ vanishes for both limits,

$$\therefore \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = \frac{m-1}{n}\int_0^1 x^n(1-x)^{m-2}\,dx.$$

The repeated application of this formula reduces the integral to depend on $\int_0^1 x^{m+n-2}\,dx$, the value of which is $\frac{1}{m+n-1}$. Hence we have

$$\int_0^1 x^{n-1}(1-x)^{m-1}\,dx = \frac{1\,.\,2\,.\,3\,.\,.\,.\,.\,(m-1)}{n\,.\,(n+1)\,.\,.\,.\,.\,(n+m-1)}.$$

This result shows that when either m or n are integer, the definite integral

$$\int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

can be easily evaluated.

When m and n are both fractional, the preceding is one of the most important definite integrals in analysis.

We purpose subsequently to give a short investigation of some of its simplest properties.

EXAMPLES.

1. $\int_0^1 x^3(1-x)^{\frac{5}{2}}\,dx.$ *Ans.* $\frac{2^5}{3\cdot 7\,.\,11\,.\,13}$.

2. $\int_0^1 x^4(1-x)^{\frac{1}{4}}\,dx.$ „ $\frac{2^{13}}{5\cdot 7\cdot 9\cdot 13\,.\,17}$.

90. **Values of** $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$ *and* $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$.

One of the simplest and most useful applications of definite integration is to the case of the circular integrals considered in the commencement of Chapter III.

We begin with the simple case of

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx.$$

If in the equation (Art. 53)

$$\int \sin^n x\, dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

we take 0 and $\frac{\pi}{2}$ for limits, the term $\frac{\cos x \sin^{n-1} x}{n}$ vanishes for both limits, and we have

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx.$$

Now, if n be an integer, the definite integral can be easily obtained: its form, however, depends on whether the index n is even or odd.

(1). Suppose the index even, and represented by $2m$, then

$$\int_0^{\frac{\pi}{2}} \sin^{2m} x\, dx = \frac{2m-1}{2m}\int_0^{\frac{\pi}{2}} \sin^{2m-2} x\, dx.$$

Similarly

$$\int_0^{\frac{\pi}{2}} \sin^{2m-2} x\, dx = \frac{2m-3}{2m-2}\int_0^{\frac{\pi}{2}} \sin^{2m-4} x\, dx,$$

and by successive applications of the formula, we get

$$\int_0^{\frac{\pi}{2}} \sin^{2m} x\, dx = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2m-1)}{2 \cdot 4 \cdot 6 \ldots\ldots\ 2m} \cdot \frac{\pi}{2}. \qquad (7)$$

(2). Suppose the index odd, and represented by $2m+1$, then

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1} x\, dx = \frac{2m}{2m+1}\int_0^{\frac{\pi}{2}} \sin^{2m-1} x\, dx.$$

Hence, it is easily seen that

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1}x\,dx = \frac{2 \cdot 4 \cdot 6 \ldots\ 2m}{3 \cdot 5 \cdot 7 \ldots (2m+1)}. \tag{8}$$

Again, it is evident from (4) that

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^n x\,dx,$$

and consequently (7) and (8) hold equally when $\cos x$ is substituted for $\sin x$.

92. **Investigation of** $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx$.

From Art. 52, when m and n are positive we have

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \frac{n-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-2} x\,dx,$$

and $$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x\,dx.$$

Hence, when one of the indices is an odd integer, the value of the definite* integral is easily found.

For, writing $2m+1$ instead of m we have

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \cos^n x\,dx = \frac{2m}{2m+n+1}\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^n x dx.$$

Hence

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1} x \cos^n x\,dx$$

$$= \frac{2m\,.\,(2m-2)\ \ldots .\ \ 2}{(2m+n+1)\,(2m+n-1)\ \ldots .\ (n+3)}\int_0^{\frac{\pi}{2}} \sin x \cos^n x\,dx$$

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* The result in this case follows also immediately from Art. 88, by making $\cos^2 x = z$; for this substitution transforms the integral into

$$\frac{1}{2}\int_0^1 (1-z)^m z^{\frac{n-1}{2}}\,dz.$$

$$= \frac{2 \cdot 4 \cdot 6 \quad \ldots \quad (2m)}{(n+1)(n+3)\ldots(n+2m+1)}. \tag{9}$$

In like manner

$$\int_0^{\frac{\pi}{2}} \sin^{2m}x \cos^{2n}x \, dx = \frac{2n-1}{2(m+n)} \int_0^{\frac{\pi}{2}} \sin^{2m}x \cos^{2n-2}x \, dx.$$

Hence

$$\int_0^{\frac{\pi}{2}} \sin^{2m}x \cos^{2n}x \, dx = \frac{1 \cdot 3 \cdot 5 \ldots \quad (2n-1)}{(2m+2) \ldots (2m+2n)} \int_0^{\frac{\pi}{2}} \sin^{2m}x \, dx$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1) \cdot 1 \cdot 3 \cdot 5 \ldots (2m-1)}{2 \cdot 4 \cdot 6 \ldots \ldots \ldots (2m+2n)} \cdot \frac{\pi}{2}. \tag{10}$$

in which m and n are supposed both positive integers.

Many elementary definite integrals are immediately reducible to one or other of the preceding forms.

For example, on making $x = \tan\theta$, we get

$$\int_0^{\infty} \frac{dx}{(1+x^2)^n} = \int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta \, d\theta = \frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{2 \cdot 4 \cdot 6 \ldots (2n-2)} \cdot \frac{\pi}{2}.$$

Similarly, by $x = a\sin\theta$, $\int_0^a x^n (a^2 - x^2)^{\frac{m}{2}} dx$, transforms into

$$a^{n+m+1} \int_0^{\frac{\pi}{2}} \sin^n\theta \cos^{m+1}\theta \, d\theta.$$

In like manner $\int_0^a (2ax - x^2)^{\frac{m}{2}} dx$,

on making $x = a(1 - \cos\theta)$, becomes

$$a^{m+1} \int_0^{\frac{\pi}{2}} \sin^{m+1}\theta \, d\theta.$$

The expressions for these integrals, when m and n are fractional in form, will be given in a subsequent Article.

EXAMPLES.

1. $\int_0^{\frac{\pi}{2}} \sin^7 x \cos^4 x \, dx.$ *Ans.* $\frac{4^2}{3 \cdot 5 \cdot 7 \cdot 11}.$

2. $\int_0^{\frac{\pi}{2}} \sin^9 x \cos^{\frac{4}{5}} \, dx.$ " $\frac{5 \cdot 10 \cdot 20 \cdot 30 \cdot 40}{9 \cdot 19 \cdot 29 \cdot 39 \cdot 49}.$

3. $\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x \, dx'$ " $\frac{1 \cdot 2 \cdot 3 \ldots (m-1)}{n \cdot (n+1) \ldots (n+m+1)}.$

4. $\int_0^1 (1 - x^2)^n \, dx.$ " $\frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)}.$

5. $\int_0^1 \frac{x^{2n} \, dx}{\sqrt{1 - x^2}}.$ " $\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \cdot \frac{\pi}{2}.$

6. $\int_0^1 \frac{x^{2n+1} \, dx}{\sqrt{1 - x^2}}.$ " $\frac{2 \cdot 4 \cdot 6 \ldots 2n}{3 \cdot 5 \cdot 7 \ldots (2n+1)}.$

7. Deduce Wallis's value for π by aid of the two preceding definite integrals.

8. $\int_0^\infty \frac{x^n dx}{(a + bx^2)^{\frac{n}{2}+1}}.$ " $\frac{2 \cdot 4 \cdot 6 \ldots (n-1)}{3 \cdot 5 \cdot 7 \ldots n} \frac{1}{\sqrt{ab^{n+1}}}.$

where n is an *odd* integer.

9. $\int_0^{2a} x^3 (2ax - x^2)^{\frac{3}{2}} \, dx.$

93. Value of $\int_0^\infty e^{-x} x^n \, dx$, when n is a positive integer.

In Art. 59 we have seen that

$$\int e^{-x} x^n dx = - e^{-x} x^n + n \int e^{-x} x^{n-1} dx.$$

Again, the expression $\frac{x^n}{e^x}$ vanishes when $x = 0$, and also when $x = \infty$, (Dif. Cal. Art. 92, Ex. 2).

Hence $$\int_0^\infty e^{-x} x^n dx = n \int_0^\infty e^{-x} x^{n-1} \, dx. \qquad (11)$$

Consequently $$\int_0^\infty e^{-x} x^n dx = 1 \cdot 2 \cdot 3 \ldots n. \qquad (12)$$

Many other forms are immediately reducible to the preceding definite integral.

For example, if we make $x = az$ we get

$$\int_0^\infty e^{-az} z^n dz = \frac{1 \cdot 2 \cdot 3 \cdots n}{a^{n+1}}.$$

in which a is supposed to be positive.

Again, to find $\int_0^1 x^m (\log x)^n \, dx$; let $x = e^{-z}$, and the integral becomes

$$(-1)^n \int_0^\infty e^{-(m+1)z} z^n \, dz = (-1)^n \frac{1 \cdot 2 \cdot 3 \cdots n}{(m+1)^{n+1}}.$$

Since $\log x = -\log \left(\frac{1}{x}\right)$, this result is often written in the form:

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{1 \cdot 2 \cdot 3 \cdots n}{(m+1)^{n+1}}. \qquad (13)$$

The definite integral $\int_0 e^{-x} x^{n-1} \, dx$, is sometimes known as The Second* Eulerian Integral, and is fundamental in the theory of definite integrals. Being obviously a function of n it is denoted by the symbol $\Gamma(n)$; and is styled the Gamma-Function.

It follows from (11) that

$$\Gamma(n+1) = n\,\Gamma(n). \qquad (14)$$

Also when n is an integer we have

$$\Gamma(n) = 1 \cdot 2 \cdot 3 \cdots n. \qquad (15)$$

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* The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$, considered in Art. 90 is, also sometimes called The First Eulerian Integral; we shall show subsequently how it can be expressed in terms of Gamma-Functions.

Again, when x is less than unity, we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \&c.$$

$$\therefore \int_0^1 \log x \frac{dx}{1-x} = \int_0^1 \log x (1 + x + x^2 + \ldots) dx$$

$$= - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) = - \frac{\pi^2}{6},$$

(by a well known result in Trigonometry).

In like manner we get

$$\int_0^1 \frac{\log x \, dx}{1+x} = - \frac{\pi^2}{8}.$$

An account of the more elementary properties of Gamma-Functions will be given at the end of this chapter.

Examples.

1. $\int_0^1 \left\{\log\left(\frac{1}{x}\right)\right\}^n dx.$ *Ans.* $1 . 2 . 3 \ldots n.$

2. $\int_0^\infty a^{-x} x^n dx.$ „ $\frac{1 . 2 \ldots n}{(\log a)^{n+1}}.$

3. $\int_0^1 dx \, . \, \frac{\log x}{1-x^2}.$ „ $\frac{\pi^2}{8}.$

4. $\int_0^1 \frac{dx \, (\log x)^{2n-1}}{1-x} = - 1 . 2 . 3 \ldots (2n-1) \left[1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots\right].$

5. $\int_0^1 \frac{dx}{x} \log\left(\frac{1+x}{1-x}\right).$ „ $\frac{\pi^2}{4}.$

94. *If u and v be both functions of x, and if v preserve the same sign while x varies from x_0 to X, then we shall have*

$$\int_{x_0}^X uv dx = U \int_{x_0}^X v dx,$$

where U is some quantity comprised between the greatest and the least values of u, between the assigned limits.

For, let A and B be the greatest and the least values of u, and we shall have, when v is positive,

$$Av > uv > Bv,$$

when v is negative,

$$Av < uv < Bv.$$

Consequently, for all values of x between x_0 and X, the expression $uvdx$ lies between $Avdx$ and $Bvdx$, and accordingly, since the sign of v does not change between the limits,

$$\int_{x_0}^{X} uvdx \text{ lies between } A\int_{x_0}^{X} vdx \text{ and } B\int_{x_0}^{X} vdx:$$

which establishes the theorem proposed.

COR. If $f(x)$ be *finite* and continuous for all values of x between the finite limits x_0 and X, then the integral

$$\int_{x_0}^{X} f(x)\,dx$$

will also have a finite value.

For, let A be the greatest value of $f(x)$, and B the least, then $\int_{x_0}^{X} f(x)\,dx$ evidently lies between the quantities

$$A\int_{x_0}^{X} dx \text{ and } B\int_{x_0}^{X} dx,$$

$$\therefore \int_{x0}^{X} f(x)\,dx > B(X - x_0) \text{ and } < A(X - x_0).$$

95. **Taylor's Theorem.**—The method of definite integration combined with that of integration by parts furnishes a simple proof of Taylor's series.

For, if in the equation

$$f(X + h) - f(X) = \int_{X}^{X+h} f'(x)\,dx$$

we assume $x = X + h - z$, we get $dx = - dz$, and also

$$\int_X^{X+h} f'(x)\,dx = \int_0^h f'(X+h-z)\,dz,$$

$$\therefore\ f(X+h) - f(X) = \int_0^h f'(X+h-z)\,dz.$$

Again, integrating by parts, we have

$$\int f'(X+h-z)\,dz = zf'(X+h-z) + \int zf''(X+h-z)\,dz.$$

Hence, substituting the limits, we have

$$\int_0^h f'(X+h-z)\,dz = hf'(X) + \int_0^h zf''(X+h-z)\,dz.$$

In like manner

$$\int zf''(X+h-z)\,dz = \frac{z^2}{2}f''(X+h-z) + \int \frac{z^2}{2}f'''(X+h-z)\,dz,$$

which gives

$$\int_0^h zf''(X+h-z)\,dz = \frac{h^2}{2}f''(X) + \int_0^h \frac{z^2}{2}f'''(X+h-z)\,dz,$$

and so on.

Accordingly, we have finally

$$f(X+h) = f(X) + \frac{h}{1}f'(X) + \frac{h^2}{1\,.\,2}f''(X) + \ldots + \frac{h^{n-1}}{\underline{|n-1}}f^{(n-1)}(X)$$

$$+ \int_0^h f^{(n)}(X+h-z)\frac{z^{n-1}dz}{\underline{|n-1}}. \tag{16}$$

This is Taylor's well-known expansion.

96. **Remainder in Taylor's Theorem expressed as a Definite Integral.**—Let R_n represent the remainder after n terms in Taylor's series, then by the preceding Article we have

$$R_n = \int_0^h f^{(n)}(X+h-z)\,\frac{z^{n-1}dz}{\underline{|n-1}}. \tag{17}$$

There is no difficulty in deducing Lagrange's form for the remainder from this result.

For, by Art. 94, we have

$$R_n = U\int_0^h \frac{z^{n-1}dz}{1 \cdot 2 \cdot 3 \ldots (n-1)} = U\frac{h^n}{1 \cdot 2 \ldots n},$$

where U lies between the greatest and least values which $f^{(n)}(X + h - z)$ assumes, while z varies between 0 and h. Hence, as in Art. 73, Dif. Cal. (since any value of z between 0 and h may be represented by $(1 - \theta)h$, where $\theta > 0$ and < 1); we have

$$R_n = \frac{h^n}{1 \cdot 2 \ldots n} f^{(n)}(X + \theta h),$$

where θ is some quantity between the limits zero and unity.

96. **Bernoulli's Series.**—If we apply the method of integration by parts to the expression $f(x)\,dx$, we get

$$\int f(x)dx = xf(x) - \int xf'(x)\,dx.$$

$$\therefore \int_0^X f(x)\,dx = Xf(X) - \int_0^X f'(x)\,xdx.$$

In like manner

$$\int_0^X f'(x)xdx = \frac{X^2}{1 \cdot 2} f'(X) - \int_0^X f''(x)\frac{x^2dx}{1 \cdot 2},$$

$$\int_0^X f''(x)\frac{x^2dx}{1 \cdot 2} = \frac{X^3}{1 \cdot 2 \cdot 3} f''(X) - \int_0^X f'''(x)\frac{x^3dx}{1 \cdot 2 \cdot 3},$$

and so on.

Hence, we get finally

$$\int_0^X f(x)\,dx = \frac{X}{1}f(X) - \frac{X^2}{1 \cdot 2}f'(X) + \frac{X^3}{1 \cdot 2 \cdot 3}f''(X) - \&c. \ldots \quad (18)$$

Compare Art. 64, Dif. Cal., where the result was obtained directly from Taylor's expansion.

98. **Exceptional Cases in Definite Integrals.**—In the foregoing discussion of definite integrals we have supposed that the function $f(x)$ under the sign of integration has a finite value for all values of x between the limits. We have also supposed that the limits are finite. We purpose now to give a short discussion of the exceptional cases.* They may be classed as follows:—(1). When $f(x)$ becomes infinite at one of the limits of integration. (2). When $f(x)$ becomes infinite for one or more values of x between the limits of integration. (3). When one or both of the limits become infinite.

In these cases, the integral $\int_{x_0}^{X} f(x)\,dx$ may still have a finite value, or it may be infinite, or indeterminate: depending on the form of the function $f(x)$ in each particular case. The following investigation will be found to comprise the cases which usually arise.

99. **Case in which $f(x)$ becomes infinite at one of the Limits.**—Suppose that $f(x)$ is finite for all values of x between x_0 and X, but that it becomes infinite when $x = X$.

The case that most commonly arises is where $f(x)$ is of the form $\frac{\psi(x)\,dx}{(X-x)^n}$, in which $\psi(x)$ is finite for all values between the limits, and n is a positive index.

Let a be assumed so that $\psi(x)$ preserves the same sign between the limits a and X, then

$$\int_{x_0}^{X} \frac{\psi(x)\,dx}{(X-x)^n} = \int_{x_0}^{a} \frac{\psi(x)\,dx}{(X-x)^n} + \int_{a}^{X} \frac{\psi(x)\,dx}{(X-x)^n}.$$

The former of the integrals at the right-hand side is finite by Art. 94. The consideration of the latter resolves into two cases, according as n is less or greater than unity.

(1). Let $n < 1$, and also let A and B be the greatest and least values of $\psi(x)$ between the limits a and X: then, by Art. 94, the integral

$$\int_{a}^{X} \frac{\psi(x)\,dx}{(X-x)^n} \text{ lies between } A\int_{a}^{X} \frac{dx}{(X-x)^n} \text{ and } B\int_{a}^{X} \frac{dx}{(X-x)^n}.$$

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* The complete investigation of definite integrals in these exceptional cases is due to Cauchy. For a more general discussion the student is referred to M. Moigno's Calcul Intégral, as also to those of M. Serret, and M. Bertrand.

Moreover, since $n < 1$, we have evidently

$$\int_a^X \frac{dx}{(X-x)^n} = \frac{(X-a)^{1-n}}{1-n},$$

and consequently, in this case, the proposed integral has a finite value.

(2). Let $n > 1$ and, as before, suppose A and B the greatest and least values of $\psi(x)$ between a and X; then

$$\int_a^X \frac{\psi(x)\,dx}{(X-x)^n} \text{ lies between } A\int_a^X \frac{dx}{(X-x)^n} \text{ and } B\int_a^X \frac{dx}{(X-x)^n}.$$

Again, we have

$$\int \frac{dx}{(X-x)^n} = \frac{-1}{(n-1)(X-x)^{n-1}}.$$

Now $\dfrac{1}{(X-x)^{n-1}}$ becomes infinite when $x = X$, but has a finite value when $x = a$; consequently the definite integral proposed has an infinite value in this case.

When $n = 1$, $\displaystyle\int \frac{dx}{(X-x)} = -\log(X-x)$, which becomes infinite when $x = X$; and consequently in this case also the integral proposed becomes infinite.

The investigation when $f(x)$ becomes infinite for $x = x_0$ follows from the preceding by interchanging the limits.

100. **Case where $f(x)$ becomes infinite between the Limits.**— Suppose $f(x)$ becomes infinite when $x = a$, where a lies between the limits x_0 and X, then since

$$\int_{x_0}^X f(x)\,dx = \int_{x_0}^a f(x)\,dx + \int_a^X f(x)\,dx$$

the investigation is reduced to two integrals, each of which may be treated as in the preceding Article.

Hence, if we suppose $f(x) = \dfrac{\psi(x)}{(x-a)^n}$, it follows, as in the last Article, that $\int_{x_0}^{X} f(x)\,dx$ has a finite or an infinite value according as n is *less* or *not less* than unity.

The case in which $f(x)$ becomes infinite for two or more values between the limits is treated in a similar manner.

For example, if

$$f(a_1) = \infty, \quad f(a_2) = \infty, \quad \ldots \, f(a_n) = \infty,$$

where $a_1, a_2 \ldots a_n$ lie between the limits X and x_0; then

$$\int_{x_0}^{X} f(x)\,dx = \int_{x_0}^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \text{\&c.} + \int_{a_n}^{X} f(x)\,dx,$$

each of which can be treated separately.

101. **Case of Infinite Limits.**—Suppose the superior limit X to be infinite, and, as in the preceding discussion, let $f(x)$ be of the form $\frac{\psi(x)}{(x-a)^n}$, where $\psi(x)$ is finite for all values of x.

As before we have

$$\int_{x_0}^{X} f(x)\,dx = \int_{x_0}^{a} f(x)\,dx + \int_{a}^{X} f(x)\,dx.$$

The integral between the finite limits x_0 and a has a finite value as before. The investigation of the other integral consists again of two cases.

(1). Let $n > 1$, and let A be the greatest value of $\psi(x)$ between the limits a and ∞, then

$$\int_{a}^{\infty} \frac{\psi(x)\,dx}{(x-a)^n} \text{ is less than } A \int_{a}^{\infty} \frac{dx}{(x-a)^n}.$$

But $$\int_{a}^{X} \frac{dx}{(x-a)^n} = \frac{1}{n-1}\left[\frac{1}{(a-a)^{n-1}} - \frac{1}{(X-a)^{n-1}}\right].$$

The latter term becomes evanescent when $X = \infty$; accordingly in this case the proposed integral has a finite value.

In like manner it is easily seen that if n be *not greater than* unity the definite integral

$$\int_a^\infty \frac{dx}{(x-a)^n}$$

has an infinite value; and consequently

$$\int_a^\infty \frac{\psi(x)\,dx}{(x-a)^n}$$

is also infinite, provided $\psi(x)$ does not become evanescent for infinite values of x.

Hence, the definite integral

$$\int_{x_0}^\infty \frac{\psi(x)\,dx}{(x-a)^n}$$

has, in general, a finite or an infinite value, according as n is greater or not greater than unity: $\psi(x)$ being supposed finite, and x_0 being greater than a.

If X become $-\infty$, a similar investigation is applicable, for on changing x into $-x$, we have

$$\int_{x_0}^{X} f(x)\,dx = -\int_{-x_0}^{-X} f(-x)\,dx,$$

in which the superior limit becomes ∞.

102. **Principal and General Values of a Definite Integral.**— We shall conclude this discussion with a short account of Cauchy's* method of investigation.

Suppose $f(x)$ to be infinite when $x = a$, where a lies between the limits x_0 and X, then the integral $\int_{x_0}^{X} f(x)\,dx$ is regarded as the limit towards which the sum

$$\int_{x_0}^{a-\mu\epsilon} f(x)\,dx + \int_{a+\nu\epsilon}^{X} f(x)\,dx$$

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* This and the four following Articles have been taken, with some modifications, from Moigno's Calcul Intégral.

approaches when ϵ becomes evanescent: μ and ν being any arbitrary constants.

This value depends on the nature of $f(x)$, and may be finite and determinate, or infinite, or indeterminate.

If we suppose $\mu = \nu$, the limiting value of the preceding sum is called the *principal value* of the proposed integral: while that given above is called its *general* value.

For example, let us consider the integral $\int_{-x_0}^{X} \frac{dx}{x}$.

Here $$\int_{-x_0}^{X} \frac{dx}{x} = \text{limit} \left[\int_{\nu\epsilon}^{X} \frac{dx}{x} + \int_{-x_0}^{-\mu\epsilon} \frac{dx}{x}\right].$$

But $$\int_{\nu\epsilon}^{X} \frac{dx}{x} = \log\left(\frac{X}{\nu\epsilon}\right).$$

Also, making $x = -z$,

$$\int_{-x_0}^{-\mu\epsilon} \frac{dx}{x} = \int_{x_0}^{\mu\epsilon} \frac{dz}{z} = \log\left(\frac{\mu\epsilon}{x_0}\right).$$

Accordingly, the principal value of $\int_{-x_0}^{X} \frac{dx}{x}$ is $\log\left(\frac{X}{x_0}\right)$, while its general value is $\log\left(\frac{X}{x_0}\right) + \log\left(\frac{\mu}{\nu}\right)$. The latter expression is perfectly arbitrary and indeterminate.

Again, let us take $\int_{-x_0}^{X} \frac{dx}{x^2}$.

As before $$\int_{-x_0}^{X} \frac{dx}{x^2} = \text{limit} \left[\int_{\nu\epsilon}^{X} \frac{dx}{x^2} + \int_{-x_0}^{-\mu\epsilon} \frac{dx}{x^2}\right].$$

But $$\int_{\nu\epsilon}^{X} \frac{dx}{x^2} = \frac{1}{\nu\epsilon} - \frac{1}{X}: \text{ and } \int_{-x_0}^{-\mu\epsilon} \frac{dx}{x^2} = \frac{1}{\mu\epsilon} - \frac{1}{x_0}.$$

$$\therefore \int_{-x_0}^{X} \frac{dx}{x^2} = \text{limit} \left[\frac{1}{\mu\epsilon} + \frac{1}{\nu\epsilon} - \frac{1}{X} - \frac{1}{x_0}\right].$$

Consequently, both the principal and the general value of the integral are infinite, in this case.

In like manner

$$\int_{-x_0}^{X} \frac{dx}{x^3} = \text{limit of } \frac{1}{2}\left(\frac{1}{\mu^2\epsilon^2} - \frac{1}{\nu^2\epsilon^2} + \frac{1}{x_0^2} - \frac{1}{X^2}\right).$$

Hence the general value of the integral is infinite, while its principal value is $\frac{1}{2}\left(\frac{1}{x_0^2} - \frac{1}{X^2}\right)$.

It may be observed that the principal value of

$$\int_{-x_0}^{X} \frac{dx}{x^3} \text{ is equal to } \int_{x_0}^{X} \frac{dx}{x^3}.$$

This holds also whenever $f(x)$ is a function of an odd order: i. e. when $f(-x) = -f(x)$.

For, we have

$$\int_{-x_0}^{x_0} f(x)\,dx = \int_0^{x_0} f(x)\,dx + \int_{-x_0}^{0} f(x)\,dx.$$

But $$\int_{-x_0}^{0} f(x)\,dx = -\int_{x_0}^{0} f(-x)\,dx = \int_0^{x_0} f(-x)\,dx.$$

$$\therefore \int_{-x_0}^{x_0} f(x)\,dx = \int_0^{x_0} \{f(x) + f(-x)\}\,dx. \tag{19}$$

Accordingly if $f(-x) = -f(x)$ we get

$$\int_{-x_0}^{x_0} f(x)\,dx = 0.$$

Again, if $f(x)$ be of an even order, i. e., if $f(-x) = f(x)$, we have

$$\int_{-x_0}^{x_0} f(x)\,dx = 2\int_0^{x_0} f(x)\,dx.$$

103. **Singular Definite Integral.**—The difference between the general and the principal value of the integral considered at the commencement of the preceding Article is represented by

$$\int_{a+\nu\epsilon}^{a+\mu\epsilon} f(x)\,dx,$$

in which $f(a) = \infty$, and ϵ is evanescent.

Such an integral is called by Cauchy a *singular definite integral*, in which the limits differ by an infinitely small quantity. The preceding discussion shows that such an integral may be either *infinite* or *indeterminate*.

104. **Infinite Limits.**—If the superior limit be infinite, we regard $\int_{x_0}^{\infty} f(x)dx$ as the limit of $\int_{x_0}^{\frac{1}{\mu\epsilon}} f(x)dx$, when ϵ becomes evanescent.

Also $\int_{-\infty}^{\infty} f(x)\,dx$ = limit of $\int_{-\frac{1}{\mu\epsilon}}^{\frac{1}{\nu\epsilon}} f(x)\,dx$, when ϵ is evanescent.

In the latter case the value of the definite integral when $\mu = \nu$ is, as before, called the *principal value* of

$$\int_{-\infty}^{\infty} f(x)dx.$$

In this we assume that $f(x)$ does not become infinite for any real value of x.

105. **Example.**—Suppose $\frac{f(x)}{F(x)}$ to be a rational algebraic fraction, in which $f(x)$ is at least two degrees lower in x than $F(x)$, and suppose all the roots of $F(x) = 0$ to be imaginary, it is required to find the value of

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx.$$

From the foregoing conditions it follows that $\frac{f(x)}{F(x)}$ cannot become infinite for any real value of x: accordingly the true value of the integral is the limit of

$$\int_{-\frac{1}{\mu\epsilon}}^{\frac{1}{\nu\epsilon}} \frac{f(x)}{F(x)}\,dx,$$

when ϵ vanishes.

To find this value, suppose $\frac{f(x)}{F(x)}$ decomposed by the method of partial fractions, and let

$$\frac{A - B\sqrt{-1}}{x - a - b\sqrt{-1}} \text{ and } \frac{A + B\sqrt{-1}}{x - a + b\sqrt{-1}}$$

be the fractions corresponding to the pair of conjugate roots

$$a + b\sqrt{-1} \text{ and } a - b\sqrt{-1}, \text{ of } F(x) = 0:$$

then the corresponding quadratic fraction is the sum of

$$\frac{A - B\sqrt{-1}}{x - a - b\sqrt{-1}} \text{ and } \frac{A + B\sqrt{-1}}{x - a + b\sqrt{-1}}$$

$$\text{i. e. } \frac{2A(x - a) + 2Bb}{(x - a)^2 + b^2}.$$

Again $$\int \frac{2Bb\,dx}{(x - a)^2 + b^2} = 2B \tan^{-1}\left(\frac{x - a}{b}\right),$$

$$\therefore \int_{-\frac{1}{\mu\epsilon}}^{\frac{1}{\nu\epsilon}} \frac{2Bb\,dx}{(x - a)^2 + b^2} = 2\pi B,$$

when ϵ vanishes.

Also $$\int \frac{2A(x - a)\,dx}{(x - a)^2 + b^2} = A \log\{(x - a)^2 + b^2\}.$$

$$\therefore \int_{-\frac{1}{\mu\epsilon}}^{\frac{1}{\nu\epsilon}} \frac{2A(x - a)\,dx}{(x - a)^2 + b^2} = A \log\left\{\frac{\mu^2}{\nu^2} \frac{(1 - a\nu\epsilon)^2 + b^2\nu^2\epsilon^2}{(1 + a\mu\epsilon)^2 + b^2\mu^2\epsilon^2}\right\}$$

$$= 2A \log\frac{\mu}{\nu}, \text{ when } \epsilon = 0.$$

$$\int_{-\frac{1}{\mu\epsilon}}^{\frac{1}{\nu\epsilon}} \frac{(2A(x-a) + 2Bb)\,dx}{(x-a)^2 + b^2} = 2A\log\left(\frac{\mu}{\nu}\right) + 2\pi B.$$

Now suppose $F(x)$ to be of the degree $2n$ in x, and let the values of A and B, corresponding to the n pairs of imaginary roots, be denoted by $A_1, A_2, \ldots A_n$, and $B_1, B_2, \ldots B_n$, respectively; then we have

$$\int_{-\frac{1}{\mu\epsilon}}^{\frac{1}{\nu\epsilon}} \frac{f(x)}{F(x)}\,dx = 2(A_1 + A_2 + \ldots + A_n)\log\left(\frac{\mu}{\nu}\right)$$
$$+ 2\pi(B_1 + B_2 + \ldots + B_n).$$

Again, since $f(x)$ is of the degree $2n-2$ at most, we have

$$A_1 + A_2 + \ldots + A_n = 0.$$

For, if we clear the equation

$$\frac{f(x)}{F(x)} = \frac{2A_1(x-a_1) + 2B_1b_1}{(x-a_1)^2 + b_1^2} + \ldots + \frac{2A_n(x-a_n) + 2B_nb_n}{(x-a_n)^2 + b_n^2}$$

from fractions, the coefficient of x^{2n-1} at the right hand side is evidently

$$2(A_1 + A_2 + \ldots + A_n);$$

which must be zero, as there is no corresponding term on the other side.

Accordingly we have, in this* case,

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx = 2\pi(B_1 + B_2 + \ldots B_n). \qquad (21)$$

---

* It may be observed that when $f(x)$ is but one degree lower than $F(x)$, the *principal value of* $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\,dx$ is still of the form given in (21).

We proceed to apply this result to an important example.

106. **Value of** $\int_0^\infty \frac{x^{2m}dx}{1+x^{2n}}$, **when** m **and** n **are positive integers, and** $n > m$.

Let a be a root of $x^{2n}+1=0$, and, by Art. 33, we have

$$A - B\sqrt{-1} = \frac{a^{2m}}{2na^{2n-1}} = -\frac{a^{2m+1}}{2n}.$$

Again, by the theory of equations, a is of the form

$$\cos\frac{(2k+1)\pi}{2n} + \sqrt{-1}\sin\frac{(2k+1)\pi}{2n},$$

in which k is a positive integer, and less than n,

$$\therefore\ a^{2m+1} = \cos(2k+1)\,\theta + \sqrt{-1}\sin(2k+1)\,\theta,$$

where
$$\theta = \frac{(2m+1)\,\pi}{2n}.$$

Hence $B = \frac{\sin(2k+1)\,\theta}{2n}$; and accordingly we have

$$B_1 + B_2 + \ldots + B_n = \frac{1}{2n}(\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\,\theta).$$

To find this sum, let

$$S = \sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\,\theta,$$

then

$$2S\sin\theta = 2\sin^2\theta + 2\sin\theta\sin 3\theta + \ldots + 2\sin\theta\sin(2n-1)\,\theta$$

$$= 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta + \ldots + \cos(2n-2)\,\theta - \cos 2n\theta$$

$$= 1 - \cos 2n\theta = 2\sin^2 n\theta = 2\sin^2(2m+1)\frac{\pi}{2} = 2.$$

$$\therefore\ S = \frac{1}{\sin\theta} = \frac{1}{\sin\frac{(2m+1)\,\pi}{2n}}.$$

Accordingly, we have

$$\int_{-\infty}^{\infty} \frac{x^{2m}dx}{1 + x^{2n}} = \frac{\pi}{n \sin \frac{(2m + 1)\pi}{2n}}.$$

Hence, by (19),

$$\int_0^\infty \frac{x^{2m}dx}{1 + x^{2n}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^{2m}dx}{1 + x^{2n}} = \frac{\pi}{2n} \cdot \frac{1}{\sin \frac{(2m + 1)\pi}{2n}}. \qquad (22)$$

We now proceed to consider the analogous integral $\int_0^\infty \frac{x^{2m}dx}{1 - x^{2n}}$, where m and n, as before, are positive integers, and $n > m$.

107. Investigation of $\int_0^\infty \frac{x^{2m}}{1 - x^{2n}}\,dx$.

We commence by showing that

$$\int_0^\infty \frac{dx}{1 - x^2} = 0.$$

This is easily seen as follows,

$$\int_0^\infty \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^\infty \frac{dx}{1 - x^2}.$$

Now, transform the latter integral, by making $x = \frac{1}{z}$, and we get

$$\int_1^\infty \frac{dx}{1 - x^2} = \int_1^0 \frac{dz}{1 - z^2} = -\int_0^1 \frac{dz}{1 - z^2} = -\int_0^1 \frac{dx}{1 - x^2}.$$

$$\therefore \int_0^\infty \frac{dx}{1 - x^2} = 0.$$

Again, proceeding to the integral

$$\int_0^\infty \frac{x^{2m}dx}{1 - x^{2n}},$$

we observe that $1 + x$ and $1 - x$ are the only real factors of $1 - x^{2n}$, and that the corresponding partial quadratic fraction in the decomposition of

$$\frac{x^{2m}}{1 - x^{2n}} \text{ is } \frac{1}{n(1 - x^2)}.$$

Consequently, the part of the definite integral which corresponds to the real roots disappears.

Moreover, it is easily seen that the method of Arts. 104 and 105 applies to the fractions arising from the $n - 1$ pairs of imaginary roots, and accordingly

$$\int_{-\infty}^{\infty} \frac{x^{2m}dx}{1 - x^{2n}} = 2\pi (B_1 + B_2 + \ldots + B_{n-1}),$$

where $B_1, B_2, \ldots B_{n-1}$ have the same signification as before.

Again, since the roots of $x^{2n} - 1 = 0$ are of the form

$$\cos\frac{k\pi}{n} \pm \sqrt{-1}\sin\frac{k\pi}{n},$$

it follows, as in Art. 105, that

$$B_1 + B_2 + \ldots + B_{n-1} = \frac{1}{2n}[\sin 2\theta + \sin 4\theta + \ldots + \sin 2(n - 1)\theta],$$

where $$\theta = \frac{(2m + 1)\pi}{2n}, \text{ as before.}$$

Proceeding as in the former case, it is easily seen that

$$\sin 2\theta + \sin 4\theta + \ldots + \sin 2(n - 1)\theta$$

$$= \frac{\cos\theta - \cos(2n - 1)\theta}{2\sin\theta} = \cot\frac{(2m + 1)\pi}{2n}.$$

Hence $$\int_{-\infty}^{\infty} \frac{x^{2m}dx}{1 - x^{2n}} = \frac{\pi}{n}\cot\frac{2m + 1}{2n}\pi.$$

$$\therefore \int_0^{\infty} \frac{x^{2m}dx}{1 - x^{2n}} = \frac{\pi}{2n}\cot\frac{2m + 1}{2n}\pi. \qquad (23)$$

Again, if we transform (22) and (23), by making $x^{2n} = z$, and $a = \frac{2m+1}{2n}$, we get

$$\int_0^\infty \frac{z^{a-1}dz}{1+z} = \frac{\pi}{\sin a\pi}, \qquad \int_0^\infty \frac{z^{a-1}dz}{1-z} = \pi \cot a\pi. \tag{24}$$

The conditions imposed on m and n require that a should be positive and less than unity.

Moreover, since the results in (24) hold for all integer values of m and n, provided $n > m$, we assume, by the law of continuity, that they hold for all values of a, so long as it is positive, and less than unity.

108. The definite integrals discussed in the two preceding articles admit of several important transformations, of which we proceed to add a few.

For example, on making $u = z^a$ in (24), we get

$$\int_0^\infty \frac{du}{1+u^{\frac{1}{a}}} = \frac{a\pi}{\sin a\pi}; \qquad \int_0^\infty \frac{du}{1-u^{\frac{1}{a}}} = a\pi \cot a\pi.$$

On making $\frac{1}{a} = r$, these become

$$\int_0^\infty \frac{du}{1+u^r} = \frac{\pi}{r\sin\frac{\pi}{r}}; \qquad \int_0^\infty \frac{du}{1-u^r} = \frac{\pi}{r}\cot\frac{\pi}{r}, \tag{25}$$

where r is positive and greater than unity.

Again

$$\int_0^\infty \frac{x^n dx}{1+x^2} = \int_0^1 \frac{x^n dx}{1+x^2} + \int_1^\infty \frac{x^n dx}{1+x^2}.$$

Now, if in the latter integral we make $x = \frac{1}{z}$, we get

$$\int_1^\infty \frac{x^n dx}{1+x^2} = -\int_1^0 \frac{z^{-n}dz}{1+z^2} = \int_0^1 \frac{x^{-n}dx}{1+x^2}$$

$$\therefore \int_0^\infty \frac{x^n dx}{1+x^2} = \int_0^1 \frac{x^n + x^{-n}}{1+x^2}dx. \tag{26}$$

Moreover, from (22), when n is less than unity, we have

$$\int_0^\infty \frac{x^n dx}{1+x^2} = \frac{\pi}{2\cos\frac{n\pi}{2}}. \tag{27}$$

Accordingly

$$\int_0^1 \frac{x^n + x^{-n}}{x + x^{-1}} \frac{dx}{x} = \frac{\pi}{2\cos\frac{n\pi}{2}}. \tag{28}$$

In like manner, it is easily seen that

$$\int_0^1 \frac{x^n - x^{-n}}{x - x^{-1}} \frac{dx}{x} = \frac{\pi}{2}\tan\frac{n\pi}{2}. \tag{29}$$

It should be noted, that in these results n must be less than unity.

Again, transform (28) and (29) by making $x = e^{-\pi z}$, and $n\pi = a$, and we get

$$\int_0^\infty \frac{e^{az} + e^{-az}}{e^{\pi z} + e^{-\pi z}} dz = \frac{\sec\frac{a}{2}}{2}, \quad \int_0^\infty \frac{e^{az} - e^{-az}}{e^{\pi z} - e^{-\pi z}} dz = \frac{\tan\frac{a}{2}}{2}. \tag{30}$$

We add a few additional examples for illustration.

EXAMPLES.

1. $\int_0^a \frac{dx}{(a^n - x^n)^{\frac{1}{n}}}$ *Ans.* $\frac{\pi}{n\sin\frac{\pi}{n}}$.

2. $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$. " $\frac{\pi}{2ab(a+b)}$.

3. $\int_0^\infty \frac{dx}{1-x^4}$. " $\frac{\pi}{4}$.

4. $\int_0^{\frac{\pi}{2}} \tan^n\theta d\theta$, where n lies between + 1 and − 1. " $\frac{\pi}{2\cos\frac{n\pi}{2}}$.

5. $\int_0^1 \frac{x^m + x^{-m}}{x^n + x^{-n}} \frac{dx}{x}$, where $n > m$. „ $\frac{\pi}{2n \cos \frac{m\pi}{2n}}$.

6. $\int_0^\infty \frac{(e^{ax} + e^{-ax})(e^{bx} + e^{-bx})}{e^{\pi x} + e^{-\pi x}} dx$. „ $\frac{\cos \frac{a}{2} \cos \frac{b}{2}}{\cos a + \cos b}$.

7. $\int_0^\infty \frac{(e^{ax} + e^{-ax})(e^{bx} - e^{-bx})}{e^{\pi x} + e^{-\pi x}} dx$. *Ans.* $\frac{\sin b}{\cos a + \cos b}$.

It should be observed, that in these we must have $a + b < \pi$.

8. Hence, when $b < \pi$, prove that

$$\int_0^\infty \frac{e^{bx} + e^{-bx}}{e^{\pi x} + e^{-\pi x}} \cos ax \, dx = \frac{\left(e^{\frac{a}{2}} + e^{-\frac{a}{2}}\right) \cos \frac{b}{2}}{e^a + 2 \cos b + e^{-a}}.$$

$$\int_0^\infty \frac{e^{bx} - e^{-bx}}{e^{\pi x} - e^{-\pi x}} \cos ax \, dx = \frac{\sin b}{e^a + 2 \cos b + e^{-a}},$$

$$\int_0^\infty \frac{e^{bx} + e^{-bx}}{e^{\pi x} - e^{-\pi x}} \sin ax \, dx = \frac{1}{2} \frac{e^a - e^{-a}}{e^a + 2 \cos b + e^{-a}}.$$

9. $\int_0^1 \frac{z^a - z^{-a}}{1 - z} dz$. „ $\pi \cot a\pi - \frac{1}{a}$.

109. **Differentiation of Definite Integrals.**—It is plain from Art. 82 that the method of *differentiation under the sign of integration* applies to definite, as well as to indefinite integrals, provided the limits of integration are independent of the quantity with respect to which we differentiate.

On account of the importance of this principle we add an independent proof, as follows:—

Suppose u to denote the definite integral in question, i. e. let

$$u = \int_a^b \phi(x, \alpha) \, dx,$$

where a and b are independent of α.

To find $\frac{du}{d\alpha}$ let Δu denote the change in u arising from the change $\Delta\alpha$ in α; then, since the limits are unaltered,

$$\Delta u = \int_a^b \{\phi(x, a + \Delta a) - \phi(x, a)\}\, dx,$$

$$\therefore \frac{\Delta u}{\Delta a} = \int_a^b \frac{\phi(x, a + \Delta a) - \phi(x, a)}{\Delta a}\, dx.$$

Hence, on passing to the limit,* we have

$$\frac{du}{da} = \int_a^b \frac{d\phi(x, a)}{da}\, dx.$$

Also, if we differentiate n times in succession, we obviously have

$$\frac{d^n u}{da^n} = \int_a^b \frac{d^n \phi(x, a)}{da^n}\, dx.$$

The importance of this method will be best exhibited by a few elementary examples.

110. **Integrals deduced by Differentiation.**—If the equation

$$\int_0^\infty e^{-ax}\, dx = \frac{1}{a}$$

be differentiated n times with respect to a, we get

$$\int_0^\infty x^n e^{-ax} dx = \frac{1 \cdot 2 \cdot 3 \ldots n}{a^{n+1}};$$

as in Art. 92.

Again, from the equation

$$\int_0^\infty \frac{dx}{x^2 + a} = \frac{\pi}{2} \frac{1}{a^{\frac{1}{2}}},$$

we get after n differentiations with respect to a,

$$\int_0^\infty \frac{dx}{(x^2 + a)^{n+1}} = \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \frac{1}{a^{n+\frac{1}{2}}},$$

which agrees with Art. 91.

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* For exceptions to this general result the student is referred to Bertrand's "Calcul Intégral," p. 181.

Again, if we take 0 and ∞ for limits in the integrals (23) and (24) of Art. 21, we get

$$\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2},\quad \int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2}. \quad (32)$$

Now, differentiate each of these n times with respect to a, and we get

$$\int_0^\infty e^{-ax}x^n\cos mx\,dx = (-1)^n\left(\frac{d}{da}\right)^n\left(\frac{a}{a^2+m^2}\right)$$

$$= \frac{\underline{|n}\,.\cos(n+1)\theta}{(a^2+m^2)^{\frac{n+1}{2}}},$$

$$\int_0^\infty e^{-ax}x^n\sin mx\,dx = \frac{\underline{|n}\,.\sin(n+1)\theta}{(a^2+m^2)^{\frac{n+1}{2}}},$$

where $m = a\tan\theta$. (See Ex. 17. 19. Dif. Cal. pp. 46, 47.)

Next, from (24) we have

$$\int_0^\infty \frac{x^{a-1}dx}{1-x} = \pi\cot a\pi.$$

Accordingly, if we differentiate with respect to a, we have

$$\int_0^\infty \frac{x^{a-1}\log x\,dx}{1-x} = -\frac{\pi^2}{\sin^2 a\pi}.$$

Again, if the equation

$$\int_0^1 y^{n-1}dy = \frac{1}{n}$$

be transformed by making $y = \dfrac{bx}{a+bx}$, it evidently gives

$$\int_0^\infty \frac{x^{m-1}\,dx}{(a+bx)^{m+1}} = \frac{1}{na\,b^n}.$$

Now, differentiating with respect to a, we have

$$\int_0^\infty \frac{x^{n-1}dx}{(a+bx)^{n+2}} = \frac{1}{n(n+1)a^2b^n}.$$

If we proceed to differentiate $m-1$ times with regard to a, we have

$$\int_0^\infty \frac{x^{n-1}dx}{(a+bx)^{m+n}} = \frac{1.2.3\ldots(m-1)}{n.(n+1)(n+2)\ldots(n+m-1)} \cdot \frac{1}{a^m b^n}.$$

111. By aid of the preceding method the determination of a definite integral can often be reduced to a known integral; we shall illustrate this statement by one or two examples.

Ex. 1. To find

$$\int_0^\pi \frac{\log(1+\sin a \cos x)}{\cos x}\,dx.$$

Denote the definite integral by u, and differentiate with respect to a, then

$$\frac{du}{da} = \int_0^\pi \frac{\cos a\, dx}{1+\sin a \cos x} = \pi, \text{ (by Art. 18).}$$

Hence, we get

$$\int_0^\pi \frac{dx \log(1+\sin a \cos x)}{\cos x} = \pi a.$$

No constant is added since the integral evidently vanishes along with a.

Ex. 2. $$u = \int_0^\infty \frac{e^{-ax}\sin mx}{x}\,dx.$$

In this case

$$\frac{du}{dm} = \int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}.$$

$$\therefore u = a\int\frac{dm}{a^2+m^2} = \tan^{-1}\left(\frac{m}{a}\right).$$

No constant is added since u vanishes with m.

Ex. 3. Next suppose

$$u = \int_0^\infty \frac{\log(1 + a^2x^2)}{1 + b^2x^2}\,dx.$$

Here

$$\frac{du}{da} = \int_0^\infty \frac{2ax^2\,dx}{(1 + a^2x^2)(1 + b^2x^2)}$$

$$= \frac{1}{a^2 - b^2}\left[\int_0^\infty \frac{2a\,dx}{1 + b^2x^2} - \int_0^\infty \frac{2a\,dx}{1 + a^2x^2}\right]$$

$$= \frac{1}{a^2 - b^2}\left(\frac{a}{b} - 1\right)\pi = \frac{\pi}{b(a+b)}.$$

$$\therefore\ u = \frac{\pi}{b}\int \frac{da}{a + b} = \frac{\pi}{b}\log(a + b) + \text{const.}$$

To determine the constant; let $a = 0$, and we obviously have $u = 0$.

Consequently, the constant is $-\dfrac{\pi}{b}\log b$.

$$\therefore\ \int_0^\infty \frac{\log(1 + a^2x^2)}{1 + b^2x^2}\,dx = \frac{\pi}{b}\log\left(\frac{a + b}{b}\right).$$

The method adopted in this article is plainly equivalent to a process of integration under the sign of integration. Before proceeding to this method we shall consider the case of differentiation when the limits a and b are functions of the quantity with respect to which we differentiate.

112. **Differentiation where the Limits are variable.**—Let the indefinite integral of the expression $\phi(x, a)\,dx$ be denoted by $F(x, a)$; then, by Art. 87, we have

$$u = \int_a^b \phi(x, a)dx = F(b, a) - F(a, a)$$

$$\therefore\ \frac{du}{db} = \frac{d.F(b, a)}{db} = \phi(b, a),$$

and

$$\frac{du}{da} = -\frac{dF(a, \alpha)}{da} = -\phi(a, \alpha).$$

Again, taking the total differential coefficient of u regarding a and b as functions of α, we have

$$\frac{du}{d\alpha} = \int_a^b \frac{d\phi(x, \alpha)}{d\alpha}dx + \frac{du}{db}\frac{db}{d\alpha} + \frac{du}{da}\frac{da}{d\alpha}$$

$$= \int_a^b \frac{d\phi(x, \alpha)}{d\alpha}dx + \phi(b, \alpha)\frac{db}{d\alpha} - \phi(a, \alpha)\frac{da}{d\alpha}. \qquad (31)$$

By repeating this process, the values of $\frac{d^2u}{d\alpha^2}$, $\frac{d^3u}{d\alpha^3}$, &c., can be obtained, if required.

113. Integration under the Sign of Integration.—Returning to the equation

$$u = \int_a^b \phi(x, \alpha)dx,$$

where the limits are independent of α; it is obvious, as in Art. 93, that

$$\int u d\alpha = \int_a^b \left[\int \phi(x, \alpha)d\alpha\right]dx,$$

provided α be taken between the same limits in both cases.

If we denote the limits of α by α_0 and α_1, we get

$$\int_{\alpha_0}^{\alpha_1} u d\alpha = \int_a^b \left[\int_{\alpha_0}^{\alpha_1} \phi(x, \alpha)d\alpha\right]dx.$$

or

$$\int_{\alpha_0}^{\alpha_1}\left[\int_a^b \phi(x, \alpha)dx\right]d\alpha = \int_a^b \left[\int_{\alpha_0}^{\alpha_1} \phi(x, \alpha)d\alpha\right]dx. \qquad (32)$$

This result is usually written in the form

$$\int_{\alpha_0}^{\alpha_1}\int_a^b \phi(x, \alpha)dx\,d\alpha = \int_a^b\int_{\alpha_0}^{\alpha_1} \phi(x, \alpha)d\alpha\,dx. \qquad (33)$$

These expressions are called *double definite integrals*, as involving successive integrations with respect to two variables, taken between limits.

It may be observed that the expression

$$\int_{a_0}^{a_1}\int_{a}^{b}\phi(x, a)\,dx\,da$$

is here taken as an abbreviation of

$$\int_{a_0}^{a_1}\left[\int_{a}^{b}\phi(x, a)\,dx\right]da,$$

in which the definite integral between the brackets is supposed to be first determined, and the result afterwards integrated with respect to a, between the limits a_0 and a_1.

The principle* established above may be otherwise stated thus, *In the determination of the integral of the differential*

$$\phi(x, a)\,dx\,da$$

between the respective limits x_0, x_1; and a_0, a_1; we may effect the integrations in either order, provided the limits of x and a are independent of each other.

In a subsequent chapter the geometrical interpretation of this, as well as of a more general theorem, will be given.

We now proceed to illustrate the importance of this method by a few examples.

114. Applications of Integration under the Sign $\int$.

Ex. 1. From the elementary equation

$$\int_0^1 x^{a-1}\,dx = \frac{1}{a}$$

we get

$$\int_0^1\int_{a_0}^{a_1} x^{a-1}\,da\,dx = \int_{a_0}^{a_1}\frac{da}{a} = \log\left(\frac{a_1}{a_0}\right).$$

---

[*] It should be noted that this principle fails whenever $\phi(x, a)$, or either of its integrals with respect to a, or to x, becomes infinite for any values of x and a contained between the limits of integration. The student will find that the examples here given are exempt from such failure.

Hence

$$\int_0^1 \frac{x^{a_1-1} - x^{a_0-1}}{\log x}\, dx = \log\left(\frac{a_1}{a_0}\right).$$

Again, if we make $x = e^{-z}$ in this equation we get

$$\int_0^\infty \frac{e^{-a_1 z} - e^{-a_0 z}}{z}\, dz = \log\left(\frac{a_1}{a_0}\right).$$

Ex. 2. We have already seen that

$$\int_0^\infty e^{-ax} \cos mx\, dx = \frac{a}{a^2 + m^2}.$$

Hence

$$\int_0^\infty \left[\int_{a_0}^{a_1} e^{-ax}\, da\right] \cos mx\, dx = \int_{a_0}^{a_1} \frac{a\,da}{a^2 + m^2}$$

$$= \tfrac{1}{2} \log\left(\frac{a_1^2 + m^2}{a_0^2 + m^2}\right).$$

or

$$\int_0^\infty \frac{e^{-a_1 x} - e^{-a_0 x}}{x} \cos mx\, dx = \tfrac{1}{2} \log\left(\frac{a_1^2 + m^2}{a_0^2 + m^2}\right).$$

Ex. 3. Again, from the equation

$$\int_0^\infty e^{-ax} \sin mx\, dx = \frac{m}{a^2 + m^2},$$

we get

$$\int_0^\infty \int_{a_0}^{a_1} e^{-ax} \sin mx\, da\, dx = \int_{a_0}^{a_1} \frac{m\,da}{a^2 + m^2}$$

$$\therefore \int_0^\infty \frac{e^{-a_0 x} - e^{-a_1 x}}{x} \sin mx\, dx = \tan^{-1}\left(\frac{a_1}{m}\right) - \tan^{-1}\left(\frac{a_0}{m}\right).$$

Compare Ex. 2, Art. 109.

If we make $a_0 = 0$ and $a_1 = \infty$ in the latter result, w obtain

$$\int_0^\infty \frac{\sin mx}{x}\, dx = \frac{\pi}{2}.$$

Ex. 4. To find the value of

$$\int_0^\infty e^{-x^2}\,dx.$$

Denoting the proposed integral by k, and substituting ax for x, we obviously have

$$\int_0^\infty e^{-a^2x^2}\,a\,dx = k.$$

$$\therefore \int_0^\infty e^{-a^2(1+x^2)}\,a\,dx = ke^{-a^2}.$$

Hence

$$\int_0^\infty \int_0^\infty e^{-a^2(1+x^2)}\,a\,da\,dx = k\int_0^\infty e^{-a^2}\,da = k^2.$$

But

$$\int_0^\infty e^{-a^2(1+x^2)}\,a\,da = \frac{1}{2}\,\frac{1}{1+x^2},$$

$$\therefore \frac{1}{2}\int_0^\infty \frac{dx}{1+x^2} = k^2, \quad \therefore \frac{\pi}{4} = k^2.$$

Hence

$$\int_0^\infty e^{-x^2}\,dx = k = \frac{1}{2}\sqrt{\pi}. \tag{34}$$

This definite integral is of considerable importance, and several others are readily deduced from it.

115. For example, to find

$$\text{(A)} \qquad u = \int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx.$$

Here

$$\frac{du}{da} = -2\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,a\,\frac{dx}{x^2}.$$

Again, let $z = \dfrac{a}{x}$, and we get

$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\frac{a\,dx}{x^2} = \int_0^\infty e^{-z^2-\frac{a^2}{z^2}}\,dz = u.$$

$$\therefore \frac{du}{da} = -2u, \text{ hence } u = Ce^{-2a}.$$

To determine C, let $a = 0$, and, by the preceding ample, u becomes $\frac{\sqrt{\pi}}{2}$.

Consequently

$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx = \frac{\sqrt{\pi}}{2}e^{-2a}.$$

Again, to find

$$\text{(B)} \qquad u = \int_0^\infty e^{-a^2x^2}\cos 2bx\,dx.$$

Here

$$\frac{du}{db} = -2\int_0^\infty e^{-a^2x^2}\sin 2bx\,x\,dx.$$

But, integrating by parts, we have

$$2\int e^{-a^2x^2}\sin 2bx\,x\,dx = -\frac{e^{-a^2x^2}\sin 2bx}{a^2} + \frac{2b}{a^2}\int e^{-a^2x^2}\cos 2bx$$

$$\therefore 2\int_0^\infty e^{a^2x^2}\sin 2bx\,x\,dx = \frac{2b}{a^2}\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx.$$

Hence

$$\frac{du}{db} = -\frac{2bu}{a^2}, \text{ or } \frac{du}{u} = -\frac{2b\,db}{a^2}.$$

Hence $$u = Ce^{-\frac{b^2}{a^2}}.$$

Also, when $b = 0$, u becomes $\frac{\sqrt{\pi}}{2a}$,

$$\therefore \int_0^\infty e^{-a^2x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}}.$$

Again, if we differentiate n times with respect to a the equation

$$\int_0^\infty e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}},$$

and afterwards make $a = 1$, we get

$$\text{(C)} \qquad \int_0^\infty e^{-x^2} x^{2n}\,dx = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^{n+1}}\sqrt{\pi}.$$

Next, to find

$$\text{(D)} \qquad \int_0^\infty \frac{\cos mx\,dx}{1+x^2}.$$

We obviously have

$$2\int_0^\infty a e^{-a^2(1+x^2)}\,da = \frac{1}{1+x^2}.$$

$$\therefore\ 2\int_0^\infty\int_0^\infty a e^{-a^2(1+x^2)}\cos mx\,dx\,da = \int_0^\infty \frac{\cos mx\,dx}{1+x^2}.$$

But, by (36), we have

$$2\int_0^\infty e^{-a^2x^2}\cos mx\,dx = \frac{\sqrt{\pi}}{a}\,e^{-\frac{m^2}{4a^2}}.$$

$$\therefore\ \sqrt{\pi}\int_0^\infty e^{-a^2-\frac{m^2}{4a^2}}\,da = \int_0^\infty \frac{\cos mx\,dx}{1+x^2}.$$

Hence, by (35), we have

$$\int_0^\infty \frac{\cos mx\,dx}{1+x^2} = \frac{\pi}{2}e^{-m}. \qquad (37)$$

Again, differentiating with respect to m, we obtain

$$\int_0^\infty \frac{x\sin mx\,dx}{1+x^2} = \frac{\pi}{2}e^{-m}. \qquad (38)$$

EXAMPLES.

1. $\int_0^\infty \frac{e^{ax}+e^{-ax}}{e^{\pi x}-e^{-\pi x}}\,x\,dx.$ *Ans.* $\frac{1}{4}\sec^2\frac{a}{2}.$

2. $\int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}\,\frac{dx}{x}.$ „ $\log\left(\tan\frac{a\pi}{2}\right).$
when $a > 0$ and < 1.

3. $\int_0^1 \frac{z^a+z^{-a}-2}{1-z}\,\frac{dz}{\log z}.$ „ $\log\left(\frac{a\pi}{\sin a\pi}\right).$

4. $\int_0^{\frac{\pi}{2}} \log(1+\cos\theta\cos x)\,\frac{dx}{\cos x}.$ „ $\frac{1}{2}\left(\frac{\pi^2}{4}-\theta^2\right).$

5. $\int_0^\infty \cos x\log\left(\frac{x^2+\beta^2}{x^2+a^2}\right)dx.$ „ $\pi\left(e^{-a}-e^{-\beta}\right).$

6. $\int_0^\infty \frac{z^r\log z\,dz}{1+z^2}.$ „ $\frac{\pi^2}{4}\,\frac{\sin\frac{r\pi}{2}}{\cos^2\frac{r\pi}{2}}.$

116. The values of some important definite integrals can be easily deduced from formula (4), Art. 88.

For example,* to find

$$\int_0^{\frac{\pi}{2}} \log(\sin\theta)\,d\theta.$$

Here $$\int_0^{\frac{\pi}{2}} \log(\sin\theta)\,d\theta = \int_0^{\frac{\pi}{2}} \log(\cos\theta)\,d\theta.$$

Hence, denoting either integral by u, we have

$$2u = \int_0^{\frac{\pi}{2}} \{\log(\sin\theta) + \log(\cos\theta)\}\,d\theta$$

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* These examples are taken from a paper, signed H. G., in the "Cambridge Mathematical Journal," Vol. 3.

$$= \int_0^{\frac{\pi}{2}} \log(\sin 2\theta)\, d\theta - \frac{\pi}{2} \log 2.$$

Again, if $z = 2\theta$, we have

$$\int_0^{\frac{\pi}{2}} \log(\sin 2\theta)\, d\theta = \tfrac{1}{2} \int_0^{\pi} \log(\sin z)\, dz$$

$$= \tfrac{1}{2} \int_0^{\frac{\pi}{2}} \log(\sin z)\, dz + \tfrac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \log(\sin z)\, dz;$$

but, since $\sin(\pi - z) = \sin z$,

$$\int_{\frac{\pi}{2}}^{\pi} \log(\sin z)\, dz = \int_0^{\frac{\pi}{2}} \log(\sin z)\, dz.$$

Consequently

$$\int_0^{\frac{\pi}{2}} \log(\sin 2\theta)\, d\theta = \int_0^{\frac{\pi}{2}} \log(\sin \theta)\, d\theta.$$

$$\therefore \int_0^{\frac{\pi}{2}} \log(\sin \theta)\, d\theta = -\frac{\pi}{2} \log(2). \qquad (39)$$

Again, to find

$$\int_0^{\pi} \theta \log(\sin \theta)\, d\theta.$$

Here

$$\int_0^{\pi} \theta \log(\sin \theta)\, d\theta = \int_0^{\pi} (\pi - \theta) \log(\sin \theta)\, d\theta$$

$$\therefore \int_0^{\pi} \theta \log(\sin \theta)\, d\theta = \frac{\pi}{2} \int_0^{\pi} \log(\sin \theta)\, d\theta = -\frac{\pi^2}{2} \log(2).$$

117. **Theorem of Frullani.**—To prove that

$$\int_0^{\infty} \frac{\phi(ax) - \phi(bx)}{x}\, dx = \phi(0) \log\left(\frac{b}{a}\right).$$

Let $u = \int_0^h \frac{\phi(z) - \phi(0)}{z}\, dz$; substitute ax for z, and we get

$$u = \int_0^{\frac{h}{a}} \frac{\phi(ax) - \phi(0)}{x}\, dx.$$

If we substitute b for a, we get

$$u = \int_0^{\frac{h}{b}} \frac{\phi(bx) - \phi(0)}{x}\, dx.$$

Hence

$$\int_0^{\frac{h}{a}} \frac{\phi(ax)\, dx}{x} - \int_0^{\frac{h}{b}} \frac{\phi(bx)\, dx}{x} = \phi(0) \int_{\frac{h}{b}}^{\frac{h}{a}} \frac{dx}{x}$$

$$= \phi(0) \log\left(\frac{b}{a}\right).$$

Or $$\int_0^{\frac{h}{a}} \frac{\phi(ax) - \phi(bx)}{x}\, dx - \int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\phi(bx)\, dx}{x} = \phi(0) \log\left(\frac{b}{a}\right). \quad (40)$$

If we suppose $h = \infty$ we get

$$\int_0^{\infty} \frac{\phi(ax) - \phi(bx)}{x}\, dx = \phi(0) \log\left(\frac{b}{a}\right), \quad (41)$$

provided $$\int_{\frac{h}{b}}^{\frac{h}{a}} \frac{\phi(bx)}{x}\, dx = 0, \text{ when } h = \infty.$$

For example, let $\phi(x) = \cos x$, and, since the integral

$$\int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\cos bx}{x}\, dx$$

evidently vanishes when $h = \infty$, we have

$$\int_0^{\infty} \frac{\cos ax - \cos bx}{x}\, dx = \log \frac{b}{a}.$$

As an exceptional case to the theorem of Frullani, let us consider the integral

$$\int_0^\infty \frac{\tan^{-1}ax - \tan^{-1}bx}{x}\,dx.$$

Since $\phi(0) = 0$ in this case, equation (40) gives

$$\int_0^\infty \frac{\tan^{-1}ax - \tan^{-1}bx}{x}\,dx = \int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\tan^{-1}bx\,dx}{x},$$

where $h = \infty$. To find the latter integral, we observe that bx is infinite for all values between the limits, and consequently $\tan^{-1}bx = \frac{\pi}{2}$, for all values of x in the definite integral.

Accordingly we have

$$\int_0^\infty \frac{\tan^{-1}ax - \tan^{-1}bx}{x}\,dx = \frac{\pi}{2}\int_{\frac{h}{a}}^{\frac{h}{b}} \frac{dx}{x} = \frac{\pi}{2}\log\left(\frac{a}{b}\right).$$

118. **Gamma-Functions.**—It may be observed that there is no branch of analysis which has occupied the attention of mathematicians more than that which treats of Definite Integrals, both single and multiple; nor in which the results arrived at are of greater elegance and interest. It would be manifestly impossible in the limits of an elementary treatise to give more than a sketch of the results arrived at. At the same time the Gamma or Eulerian Integrals hold so fundamental a place, that no treatise, however elementary, would be complete without giving at least an outline of their properties. With such an outline we propose to conclude this chapter.

The definitions of the Eulerian Integrals, both First and Second, have been given already in Art. 92.

The First Eulerian Integral, viz.:

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$$

is evidently a function of its two parameters, m and n; it is usually represented by the notation, $B(m, n)$.

Thus, we have by definition

$$\left.\begin{aligned}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx &= B(m,n).\\ \int_0^\infty e^{-x}x^{p-1}\,dx &= \Gamma(p).\end{aligned}\right\}\quad(42)$$

The constants m, n, are supposed *positive* in all cases.

It is evident that the result in equation (14), Art. 92, still holds when p is of fractional form.

Hence, we have in all cases

$$\Gamma(p+1) = p\Gamma(p).$$

This may be regarded as the fundamental property of Gamma-Functions, and by aid of it the calculations of all such functions can be reduced to those for which the parameter p is comprised between any two consecutive integers. For this purpose the values of $\Gamma(p)$, or rather of $\log\Gamma(p)$ have been tabulated by Legendre* to 12 decimal places, for all values of p (between 1 and 2) to 3 decimal places. The student will find tables to 6 decimal places at the end of this chapter. By aid of such tables we can readily calculate the approximate values of all definite integrals which are reducible to Gamma-Functions.

It may be remarked that we have

$$\Gamma(1) = 1,\quad \Gamma(0) = \infty,\quad \Gamma(-p) = \infty,$$

p being any integer. For negative values of p which are not integer the function has a finite value.

Again, if we substitute zx instead of x, where z is a constant with respect to x, we obviously have

$$\int_0^\infty e^{-xz}x^{m-1}\,dx = \frac{\Gamma(m)}{z^m}. \qquad (43)$$

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* See Traité des Fonctions Elliptiques, Tome 2. Int. Euler. chap. 16.

With respect to the First Eulerian Integral, we have already seen (Art. 90) that

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx.$$

$$\therefore\ B(m, n) = B(n, m).$$

Hence, the interchange of the constants m and n does not alter the value of the integral.

Again, if we substitute $\frac{y}{1+y}$ for x, we get

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{y^{m-1}\,dy}{(1+y)^{m+n}}.$$

Hence
$$\int_0^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}} = B(m, n). \tag{44}$$

We now proceed to express $B(m, n)$ in terms of Gamma-Functions.

119. To prove that

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

From equation (43) we have

$$\Gamma(m) = \int_0^\infty e^{-zx}\,z^m\,x^{m-1}\,dx.$$

Hence

$$\Gamma(m)\,e^{-z}\,z^{n-1} = \int_0^\infty e^{-z(1+x)}\,z^{m+n-1}\,x^{m-1}\,dx.$$

$$\therefore\ \Gamma(m)\int_0^\infty e^{-z}\,z^{n-1}dz = \int_0^\infty\left[\int_0^\infty e^{-z(1+x)}\,z^{m+n-1}\,dz\right]x^{m-1}dx.$$

But, if $z(1+x)=y$, we get

$$\int_0^\infty e^{-z(1+x)} z^{m+n-1}\,dz = \frac{1}{(1+x)^{m+n}}\int_0^\infty e^{-y} y^{m+n-1}\,dy$$

$$= \frac{\Gamma(m+n)}{(1+x)^{m+n}}.$$

$$\therefore\ \Gamma(m)\,\Gamma(n) = \Gamma(m+n)\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}.$$

Accordingly, by (44), we have

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}. \tag{45}$$

Again, if $m = 1 - n$, we get, by (24),

$$\Gamma(n)\,\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}\,dx}{1+x} = \frac{\pi}{\sin n\pi}. \tag{46}$$

If in this $n=\frac{1}{2}$, we get

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

This agrees with (34), for if we make $x^2 = z$, we get

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-z} z^{-\frac{1}{2}}\,dz = \frac{\Gamma(\frac{1}{2})}{2}.$$

120. By aid of the relation in (45) a number of definite integrals are reducible to gamma-functions.

For instance, we have

$$\int_0^\infty \frac{y^{m-1}\,dy}{(1+y)^{m+n}} = \int_0^1 \frac{y^{m-1}\,dy}{(1+y)^{m+n}} + \int_1^\infty \frac{y^{m-1}\,dy}{(1+y)^{m+n}},$$

now, substituting $\frac{1}{x}$ for y in the last integral, we get

$$\int_1^\infty \frac{y^{m-1}\,dy}{(1+y)^{m+n}} = \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}}.$$

Hence

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1 + x)^{m+n}}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m + n)}. \tag{47}$$

Next, if we make $x = \frac{ay}{b}$, we get

$$\int_0^\infty \frac{x^{m-1}dx}{(1 + x)^{m+n}} = a^m b^n \int_0^\infty \frac{y^{m-1}\,dy}{(ay + b)^{m+n}}.$$

$$\therefore \int_0^\infty \frac{y^{m-1}\,dy}{(ay + b)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{a^m b^n\,\Gamma(m + n)}. \tag{48}$$

Again,* let $x = \sin^2\theta$, and we get

$$\int_0^1 x^{m-1}(1 - x)^{n-1}\,dx = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta d\theta.$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,\cos^{2n-1}\theta d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m + n)}. \tag{49}$$

This result may also be written as follows :

$$\int_0^{\frac{\pi}{2}} \sin^{p-1}\theta\,\cos^{q-1}\theta\,d\theta = \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}{2\Gamma\left(\frac{p+q}{2}\right)}. \tag{50}$$

If we make $q = 1$, we get

$$\int_0^{\frac{\pi}{2}} \sin^{p-1}\theta\,d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}. \tag{51}$$

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* These results may be regarded as generalizations of the formulæ given in Arts. 91, 92, to which the student can readily see that they are reducible when the indices are integers.

Again, if $p = q$ in (50) it becomes

$$\frac{\left\{\Gamma\left(\frac{p}{2}\right)\right\}^2}{2\,\Gamma(p)} = \int_0^{\frac{\pi}{2}} \sin^{p-1}\theta \cos^{p-1}\theta\, d\theta = \frac{1}{2^{p-1}}\int_0^{\frac{\pi}{2}} \sin^{p-1} 2\theta\, d\theta.$$

Let $2\theta = z$, and we have

$$\int_0^{\frac{\pi}{2}} \sin^{p-1} 2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi} \sin^{p-1} z\, dz = \int_0^{\frac{\pi}{2}} \sin^{p-1} z\, dz$$

$$= \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}.$$

Hence $$\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^{p-1}}\,\Gamma(p).$$

If we substitute $2m$ for p, this becomes

$$\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m). \tag{52}$$

Again, make $y = \tan^2\theta$ in (48), and we get

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{2a^m b^n\,\Gamma(m+n)}. \tag{53}$$

121. **To find the value* of**

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right),$$

n being any integer.

Multiply the expression by itself, reversing the order of the factors, and we get its square under the form

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* This important theorem is due to Euler, by whom, as already noticed, the Gamma-functions were first investigated.

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right),$$

that is, by (46),

$$\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}}.$$

To calculate this expression, we have by the theory of equations

$$\frac{1-x^{2n}}{1-x^2}=$$

$$\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots\left(1-2x\cos\frac{(n-1)\pi}{n}+x^2\right).$$

Making successively in this, $x=1$, and $x=-1$, and replacing the first member by its true value n, we get

$$n=\left(2\sin\frac{\pi}{2n}\right)^2\left(2\sin\frac{2\pi}{2n}\right)^2\ldots\left(2\sin\frac{(n-1)\pi}{2n}\right)^2$$

$$n=\left(2\cos\frac{\pi}{2n}\right)^2\left(2\cos\frac{2\pi}{2n}\right)^2\ldots\left(2\cos\frac{(n-1)\pi}{2n}\right)^2,$$

whence, multiplying and extracting the square root,

$$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}.$$

Hence, it follows that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}. \qquad (54)$$

122. To find the value of

$$\int_0^\infty e^{-ax} \cos bx\, x^{m-1}\, dx, \quad \int_0^\infty e^{-ax} \sin bx\, x^{m-1}\, dx.$$

If in (43), $a - b\sqrt{-1}$ be substituted* for z, it becomes

$$\int_0^\infty e^{-ax} e^{bx\sqrt{-1}} x^{m-1}\, dx = \frac{\Gamma(m)}{(a - b\sqrt{-1})^m} = \frac{\Gamma(m)\,(a + b\sqrt{-1})^m}{(a^2 + b^2)^m}.$$

Let $a = (a^2 + b^2)^{\frac{1}{2}} \cos\theta$, then $b = (a^2 + b^2)^{\frac{1}{2}} \sin\theta$; and the preceding result becomes

$$\int_0^\infty e^{-ax} (\cos bx + \sqrt{-1}\sin bx)\, x^{m-1}\, dx$$

$$= \frac{\Gamma(m)}{(a + b^2)^{\frac{m}{2}}} (\cos\theta + \sqrt{-1}\sin\theta)^m$$

$$= \frac{\Gamma(m)}{(a^2 + b^2)^{\frac{m}{2}}} (\cos m\theta + \sqrt{-1}\sin m\theta).$$

Hence, equating real and imaginary parts, we have

$$\left.\begin{aligned} \int_0^\infty e^{-ax} \cos bx\, x^{m-1}\, dx &= \frac{\Gamma(m)}{(a^2 + b^2)^{\frac{m}{2}}} \cos m\theta \\ \int_0^\infty e^{-ax} \sin bx\, x^{m-1}\, dx &= \frac{\Gamma(m)}{(a^2 + b^2)^{\frac{m}{2}}} \sin m\theta \end{aligned}\right\}, \qquad (55)$$

in which $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$.

If we make $a = 0$, θ becomes $\dfrac{\pi}{2}$, and these formulæ become

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* For a rigid proof of the validity of this transformation the student is referred to Serrett's Cal. Int., p. 194.

$$\left.\begin{aligned}\int_0^\infty \cos bx\, x^{m-1}\, dx &= \frac{\Gamma(m)}{b^m}\cos\frac{m\pi}{2},\\ \int_0^\infty \sin bx\, x^{m-1}\, dx &= \frac{\Gamma(m)}{b^m}\sin\frac{m\pi}{2}.\end{aligned}\right\}. \qquad (56)$$

It may be observed that these latter integrals can be arrived at in another manner, as follows.

From (43) we have

$$\Gamma(n)\frac{\cos bz}{z^n} = \int_0^\infty e^{-zx} x^{n-1} \cos bz\, dx.$$

$$\therefore\ \Gamma(n)\int_0^\infty \frac{\cos bz\, dz}{z^n} = \int_0^\infty\int_0^\infty e^{-zx} \cos bz\, x^{n-1}\, dx\, dz.$$

But, by (32), we have

$$\int_0^\infty e^{-xz} \cos bz\, dz = \frac{x}{b^2+x^2},$$

$$\therefore \int_0^\infty \frac{\cos bz\, dz}{z^n} = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^n dx}{b^2+x^2}$$

$$= \frac{b^{n-1}}{\Gamma(n)}\,\frac{\pi}{2\cos\frac{n\pi}{2}}, \text{ by } (27),$$

in which n must be positive and < 1.

In like manner we find

$$\int_0^\infty \frac{\sin bz\, dz}{z^n} = \frac{b^{n-1}}{\Gamma(n)}\,\frac{\pi}{2\sin\frac{n\pi}{2}}.$$

The results in (56) follow from these by aid of the relation contained in equation (46).

EXAMPLES.

1. $\int_0^1 x^m (1 - x^n)^p \, dx.$ *Ans.* $\dfrac{\Gamma(p+1)\,\Gamma\left(\dfrac{m+1}{n}\right)}{\Gamma\left(p+1+\dfrac{m+n}{n}\right)}.$

2. $\int_0^1 \dfrac{x^{m-1}(1-x)^{n-1}dx}{(a+x)^{m+n}}.$,, $\dfrac{\Gamma(m)\,\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}.$

3. Prove that

$$\int_0^1 \frac{x^2dx}{(1-x^4)^{\frac{1}{2}}} \times \int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \frac{\pi}{4}.$$

4. $\int_0^\infty \cos(bz^{\frac{1}{n}})\, dz.$,, $\dfrac{\Gamma(n+1)\cos\left(b\dfrac{\pi}{2}\right)}{b^n}.$

123. **Numerical Calculation of Gamma-Functions.**—The following Table gives the values of log $\Gamma(p)$, to six decimal places, for all values of p between 1 and 2, (taken to three decimal places).

It may be observed that we have $\Gamma(1) = 1$, $\Gamma(2) = 1$, and that for all values of p between 1 and 2, $\Gamma(p)$ is positive and less than unity; and hence the values of log $\Gamma(p)$ are negative for all such values. Consequently, as in ordinary trigonometrical logarithmic Tables, the Tabular logarithm is obtained by adding 10 to the natural logarithm. The method of calculating these Tables is too complicated for insertion in an elementary Treatise.

p	0	1	2	3	4	5	6	7	8	9
1.00		9750	9500	9251	9003	8755	8509	8263	8017	7773
1.01	9.997529	7285	7043	6801	6560	6320	6080	5841	5602	5365
1.02	5128	4892	4656	4421	4187	3953	3721	3489	3257	3026
1.03	2796	2567	2338	2110	1883	1656	1430	1205	0981	0775
1.04	0533	0311	0089	$\overline{9868}$	$\overline{9647}$	9427	$\overline{9208}$	8989	$\overline{8772}$	8554
1.05	9.988338	8122	7907	7692	7478	7265	7052	6841	6629	6419
1.06	6209	6000	5791	5583	5378	5169	4963	4758	4553	4349
1.07	4145	3943	3741	3539	3338	3138	2939	2740	2541	2344
1.08	2147	1951	1755	1560	1365	1172	0978	0786	0594	0403
1.09	0212	0022	$\overline{9833}$	9644	9456	$\overline{9269}$	9082	$\overline{8900}$	$\overline{8710}$	$\overline{8525}$
1.10	9.978341	8157	7974	7791	7610	7428	7248	7068	6888	6709
1.11	6531	6354	6177	6000	5825	5650	5475	5301	5128	4955
1.12	4783	4612	4441	4271	4101	3932	3764	3596	3429	3262
1.13	3096	2931	2766	2602	2438	2275	2113	1951	1790	1629
1.14	1469	1309	1150	0992	0835	0677	0521	0365	0210	0055
1.15	9.969901	9747	9594	9442	9290	9139	8988	8838	8688	8539
1.16	8390	8243	8096	7949	7803	7658	7513	7369	7225	7082
1.17	6939	6797	6655	6514	6374	6234	6095	5957	5818	5681
1.18	5544	5408	5272	5137	5002	4868	4734	4601	4469	4337
1.19	4205	4075	3944	3815	3686	3557	3429	3302	3175	3048
1.20	2922	2797	2672	2548	2425	2302	2179	2057	1936	1815
1.21	1695	1575	1456	1337	1219	1101	0984	0867	0751	063
1.22	0521	0407	0293	0180	0067	$\overline{9955}$	$\overline{9843}$	$\overline{9732}$	$\overline{9621}$	$\overline{9511}$
1.23	9.959401	9292	9184	9076	8968	8861	8755	8649	8544	8439
1.24	8335	8231	8128	8025	7923	7821	7720	7620	7520	7420
1.25	7321	7223	7125	7027	6930	6834	6738	6642	6547	6453
1.26	6359	6267	6173	6081	5989	5898	5807	5716	5627	5537
1.27	5449	5360	5273	5185	5099	5013	4927	4842	4757	4673
1.28	4589	4506	4423	4341	4259	4178	4097	4017	3938	3858
1.29	3780	3702	3624	3547	3470	3394	3318	3243	3168	3094
1.30	3020	2947	2874	2802	2730	2659	2588	2518	2448	2379
1.31	2310	2242	2174	2106	2040	1973	1907	1842	1777	1712
1.32	1648	1585	1522	1459	1397	1336	1275	1214	1154	1094
1.33	1035	0977	0918	0861	0803	0747	0690	0634	0579	0524
1.34	0470	0416	0362	0309	0257	0205	0153	0102	0051	0001
1.35	9.949951	9902	9853	9805	9757	9710	9663	9617	9571	9525
1.36	9480	9435	9391	9348	9304	9262	9219	9178	9136	9095
1.37	9054	9015	8975	8936	8898	8859	8822	8785	8748	8711
1.38	8676	8640	8605	8571	8537	8503	8470	8437	8405	8373
1.39	8342	8311	8280	8250	8221	8192	8163	8135	8107	8080
1.40	8053	8026	8000	7975	7950	7925	7901	7877	7854	7831
1.40	7808	7786	7765	7744	7723	7703	7683	7664	7645	7626
1.42	7608	7590	7573	7556	7540	7524	7509	7494	7479	7465
1.43	7451	7438	7425	7413	7401	7389	7378	7368	7357	7348
1.44	7338	7329	7321	7312	7305	7298	7291	7284	7278	7273
1.45	7262	7263	7259	7255	7251	7248	7246	7244	7242	7241
1.46	7240	7239	7239	7240	7240	7242	7243	7245	7248	7251
1.47	7254	7258	7262	7266	7271	7277	7282	7289	7295	7302
1.48	7310	7317	7326	7334	7343	7353	7363	7373	7384	7395
1.49	7407	7419	7431	7444	7457	7471	7485	7499	7514	7529

Log Γ(*p*).

p	0	1	2	3	4	5	6	7	8
1.50	9.947545	7561	7577	7594	7612	7629	7647	7666	768
1.51	7724	7744	7764	7785	7806	7828	7850	7873	789
1.52	7943	7967	7991	8016	8041	8067	8093	8120	814
1.53	8201	8229	8258	8287	8316	8346	8376	8406	843
1.54	8500	8532	8564	8597	8630	8664	8698	8732	876
1.55	8837	8873	8910	8946	8983	9021	9059	9097	913
1.56	9214	9254	9294	9334	9375	9417	9458	9500	954
1.57	9629	9672	9716	9761	9806	9851	9896	9942	998
1.58	9.950082	0130	0177	0225	0274	0323	0372	0422	047
1.59	0573	0624	0676	0728	0780	0833	0886	0939	099
1.60	1102	1157	1212	1268	1324	1380	1437	1494	155
1.61	1668	1727	1786	1845	1905	1965	2025	2086	214
1.62	2271	2333	2396	2459	2522	2586	2650	2715	278
1.63	2911	2977	3043	3110	3177	3244	3312	3380	344
1.64	3587	3656	3726	3797	3867	3938	4010	4081	415
1.65	4299	4372	4446	4519	4594	4668	4743	4819	489
1.66	5047	5124	5201	5278	5356	5434	5513	5592	567
1.67	5830	5911	5991	6072	6154	6235	6317	6400	648
1.68	6649	6733	6817	6901	6986	7072	7157	7243	732
1.69	7503	7590	7678	7766	7854	7943	8032	8122	821
1.70	8391	8482	8573	8664	8756	8848	8941	9034	912
1.71	9314	9409	9502	9598	9693	9788	9884	9980	007
1.72	9.960271	0369	0467	0565	0664	0763	0862	0961	106
1.73	1262	1363	1464	1566	1668	1770	1873	1976	207
1.74	2287	2391	2496	2601	2706	2812	2918	3024	313
1.75	3345	3453	3561	3669	3778	3887	3996	4105	421
1.76	4436	4547	4659	4770	4882	4994	5107	5220	533
1.77	5561	5675	5789	5904	6019	6135	6251	6367	648
1.78	6718	6835	6953	7071	7189	7308	7427	7547	766
1.79	7907	8023	8149	8270	8392	8514	8636	8759	888
1.80	9129	9253	9377	9501	9626	9751	9877	0008	012
1.81	9.970383	0509	0637	0765	0893	1021	1150	1279	140
1.82	1668	1798	1929	2060	2191	2322	2454	2586	271
1.83	2985	3118	3252	3386	3520	3655	3790	3925	406
1.84	4333	4470	4606	4744	4881	5019	5157	5295	543
1.85	5712	5852	5992	6132	6273	6414	6555	6697	683
1.86	7123	7266	7408	7552	7696	7840	7984	8128	827
1.87	8564	8710	8856	9002	9149	9296	9443	9591	973
1.88	9.980036	0184	0333	0483	0633	0783	0933	1084	123
1.89	1537	1689	1841	1994	2147	2299	2453	2607	276
1.90	3069	3224	3379	3535	3690	3846	4003	4159	431
1.91	4631	4789	4947	5105	5264	5423	5582	5742	590
1.92	6223	6383	6544	6706	6867	7029	7192	7354	751
1.93	7844	8007	8171	8336	8500	8665	8830	8996	916
1.94	9494	9660	9827	9995	0162	0330	0498	0666	083
1.95	9.991173	1343	1512	1683	1853	2024	2195	2366	253
1.96	2881	3054	3227	3399	3573	3746	3920	4094	426
1.97	4618	4794	4969	5145	5321	5498	5674	5851	602
1.98	6384	6562	6740	6919	7098	7277	7457	7637	781
1.99	8178	8359	8540	8722	8903	9085	9268	9450	963

EXAMPLES.

1. $\int_0^a \frac{dx}{a^2 + x^2}$, $\int_0^\infty \frac{dx}{a^2 + x^2}$. *Ans.* $\frac{\pi}{4a}$, $\frac{\pi}{2a}$.

2. $\int_{-\infty}^{\infty} \frac{dx}{a + 2bx + cx^2}$, $ac - b^2$ being positive. ,, $\frac{\pi}{\sqrt{ac - b^2}}$.

3. Prove that

$$\int_0^\infty \frac{dx}{a + 2bx^2 + cx^4} = \frac{\pi}{2\sqrt{ah}}, \text{ where } h = 2\left(\sqrt{ac} + b\right).$$

4. $\int_0^\pi \frac{dx}{1 + \cos\theta \cos x}$. ,, $\frac{\pi}{\sin\theta}$.

5. $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos\theta \cos x}$. ,, $\frac{\pi}{2} \frac{\theta}{\sin\theta}$.

6. $\int_0^{\frac{\pi}{2}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x}$. ,, $\frac{\pi}{2ab}$.

7. $\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2}$. ,, $\frac{\pi(a^2 + b^2)}{4a^3 b^3}$.

8. $\int_0^a \sqrt{a^2 - x^2} \cos^{-1}\frac{x}{a}\, dx$. ,, $\frac{\pi^2}{8} + \frac{1}{4}$.

9. $\int_{-1}^{+1} \frac{dx}{(a - bx)\sqrt{1 - x^2}}$ $a > b$. ,, $\frac{\pi}{\sqrt{a^2 - b^2}}$.

10. $\int_a^\beta \frac{dx}{\sqrt{(x - a)(\beta - x)}}$. ,, π.

11. $\int_{a - \sqrt{a^2 - b^2}}^{a + \sqrt{a^2 - b^2}} \frac{(y^2 + b^2)\, y\, dy}{\sqrt{4a^2y^2 - (y^2 + b^2)^2}}$. ,, πa^2.

12. Show that $\int_0^\infty \frac{\sin ax \cos bx}{x}\, dx = \frac{\pi}{2}$ or 0, according as $a >$ or $< b$; and that when $a = b$ the value of the integral is $\frac{\pi}{4}$.

13. $\int_{-1}^{+1} \frac{dx}{\sqrt{(1 - 2ax + a^2)(1 - 2bx + b^2)}}$, $ab < 1$. *Ans.* $\frac{1}{ab} \log\left(\frac{1 + \sqrt{ab}}{1 - \sqrt{ab}}\right)$.

14. $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$. „ $\frac{1}{2}(\log 2 - \frac{1}{2})$.

15. $\int_0^{\frac{3\pi}{4}} \frac{\sin x dx}{1 + \cos^2 x}$. „ $\frac{\pi}{4} + \tan^{-1}\frac{1}{\sqrt{2}}$.

16. If every infinitesimal element of the side c of any triangle be divided by its distance from the opposite angle C, and the sum taken, show that its value is

$$\log\left(\cot\frac{A}{2} \cot\frac{B}{2}\right).$$

17. Being given the base of a triangle; if the sum of every element of the base multiplied by the square of its distance from the vertex be constant, show that the locus of the vertex is an ellipse.

18. $\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta \sin\theta \, d\theta}{1 + e^2 \cos^2\theta}$. *Ans.* $\frac{1}{e^2} - \frac{\tan^{-1} e}{e^3}$.

19. $\int_0^{\frac{\pi}{2}} \frac{\cos^2\theta \sin\theta \, d\theta}{\sqrt{1 + e^2 \cos^2\theta}}$. „ $\frac{(1 + e^2)\tan^{-1} e}{e^2} - \frac{1}{e}$.

20. Deduce the expansions for $\sin x$ and $\cos x$ from Bernoulli's series.

21. Show that the integral

$$\int_0^1 x^n (\log x)^m dx$$

can be immediately evaluated by the method of Art. 109, when m is an integer.

22. $\int_0^{\infty} \frac{\tan^{-1}(ax) \, dx}{x(1 + x^2)}$. *Ans.* $\frac{\pi}{2} \log(1 + a)$.

23. Find the value of

$$\int_0^{\pi} \log(1 - 2a \cos x + a^2) \, dx,$$

distinguishing between the cases where a is >, or < 1.

Ans. $a < 1$, its value is 0.

„ $a > 1$, its value is $2 \log a$.

24. If $f(x)$ can be expanded in a series of the form

$$a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots,$$

show that any coefficient after a_0 can be exhibited in the form of a definite integral.

Ans. $a_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x) \cos nx \, dx.$

25. Find the analogous theorem when $f(x)$ can be expanded in a series of sines of multiples of x; and apply the method to prove the relation

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \&c.\right),$$

when x lies between $\pm\, \pi$.

26. Prove the identical relation

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \pi.$$

27. Express the definite integral

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2\theta}}$$

in the form of a series, κ being < 1.

Ans. $\dfrac{\pi}{2}\left(1 + \left(\dfrac{1}{2}\right)^2 \kappa^2 + \left(\dfrac{1 \cdot 3}{2 \cdot 4}\right)^2 \kappa^4 + \left(\dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \kappa^6 + \&c.\right)$

28. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\log(1 + \cos a \cos x)\, dx}{\cos x}.$ *Ans.* $\dfrac{1}{2}\left(\dfrac{\pi^2}{4} - a^2\right).$

29. $\displaystyle\int_0^{\infty} x e^{-ax} \sin bx \, dx$, where $a > 0$. ,, $\dfrac{a^2 - b^2}{(a^2 + b^2)^2}.$

30. $\displaystyle\int_0^{\infty} \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2}\, dx.$,, $\dfrac{\pi}{2} \log \left\{\dfrac{(\alpha + \beta)^{\alpha+\beta}}{\alpha^{\alpha} \beta^{\beta}}\right\}.$

31. $\displaystyle\int_0^{\frac{\pi}{2}} \log(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta)\, d\theta.$,, $\pi \log \dfrac{\alpha + \beta}{2}.$

32. $\displaystyle\int_0^{\frac{\pi}{2}} \log\left(\frac{a + b\sin\theta}{a - b\sin\theta}\right) \frac{d\theta}{\sin\theta},\quad a > b.$,, $\pi \sin^{-1}\left(\dfrac{b}{a}\right).$

33. $\displaystyle\int_0^1 \frac{dx}{(1 - x^6)^{\frac{1}{6}}}$. *Ans.* $\dfrac{\pi}{3}$.

34. $\displaystyle\int_0^1 \frac{dx}{(1 - x)^{\frac{1}{n}}}$. " $\dfrac{\pi}{n \sin \dfrac{\pi}{n}}$.

35. $\displaystyle\int_0^\pi \frac{\cos rx\, dx}{1 - 2a \cos x + a^2}$. " $\dfrac{\pi a^r}{1 - a^2}$.

36. Find the sum of the series

$$\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \frac{n}{n^2 + 3^2} + \ldots + \frac{n}{2n^2}.$$

when n is increased indefinitely.

This is evidently represented by the definite integral

$$\int_0^1 \frac{dx}{1 + x^2}, \text{ or } = \frac{\pi}{4}.$$

37. Find the limit of the sum

$$\frac{1}{\sqrt{n^2 - 1^2}} + \frac{1}{\sqrt{n^2 - 2^2}} + \frac{1}{\sqrt{n^2 - 3^2}} + \ldots + \frac{1}{\sqrt{n^2 - (n - 1)^2}},$$

when $n = \infty$. *Ans.* $\dfrac{\pi}{2}$.

38. Prove that

$$\int_0^{\frac{\pi}{2}} \cos^m x \cos nx\, dx = \frac{m(m - 1)}{m^2 - n^2} \int_0^{\frac{\pi}{2}} \cos^{m-2} x \cos nx\, dx;$$

and hence, deduce the values of the integrals

$$\int_0^{\frac{\pi}{2}} \cos^{2m} x \cos(2n + 1)x\, dx, \text{ and } \int_0^{\frac{\pi}{2}} \cos^{2m+1} x \cos 2nx\, dx,$$

when m and n are integers.

39. $\displaystyle\int_0^\pi \log(1 - 2a \cos\theta + a^2) \cos n\theta\, d\theta$, when $a^2 < 1$. " $\dfrac{\pi}{2}\dfrac{a^n}{n}$

40. $\displaystyle\int_{-\infty}^{\infty} \cos \frac{\pi x^2}{2}\, dx$. " 1.

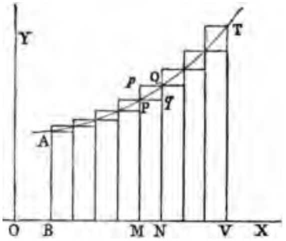
CHAPTER VII.

AREAS OF PLANE CURVES.

124. **Areas of Curves.**—The simplest method of regarding the area of a curve is by supposing it referred to rectangular axes of co-ordinates; then, the area included between the curve, the axis of x, and two ordinates corresponding to the values x_0 and x_1 of x, is represented by the definite integral

$$\int_{x_0}^{x_1} y\,dx.$$

For, let the area in question be represented by the space $ABVT$, and suppose BV divided into n equal intervals, and the corresponding ordinates drawn, as in the accompanying figure.



Then the area of the portion $PMNQ$ is less than the rectangle $pMNQ$, and greater than $PMNq$.

Hence the entire area $ABVT$ is less than the sum of the rectangles represented by $pMNQ$, and greater than the sum of the rectangles $PMNq$; but the difference between these latter sums is the sum of the rectangles $Pp\,Qq$, or (since the rectangles have equal bases) the rectangle under MN and the difference between TV and AB. Now, by supposing the number n increased indefinitely, MN can be made indefinitely small, and hence the rectangle $MN\,(TV \sim AB)$ also becomes infinitely small. Consequently the difference between the area $ABVT$ and the sum of the rectangles $PMNq$ becomes evanescent at the same time.

If now the co-ordinates of P be denoted by x and y, and MN by Δx, it follows that the area $ABVT$ is the limiting value* of $\Sigma (y \Delta x)$, when the increment Δx becomes infinitely small.

Or area $ABVT = \int_{x_0}^{x_1} y dx$; where $x_1 = OV$, $x_0 = OB$.

It should be observed that this result requires that y continues finite, and of the same sign, between the limits of integration.

If y change its sign between the limits, i. e. if the curve cut the axis of x, the preceding definite integral represents the difference of the areas at opposite sides of the axis of x.

In such cases it is preferable to consider each area separately, by dividing the integral into two parts, separated by the value of x for which y vanishes.

The preceding mode of proof obviously applies also to the case where the co-ordinate axes are oblique; in which case the area is represented by

$$\sin \omega \int_{x_0}^{x_1} y dx$$

where ω represents the angle between the axes.

In applying these formulæ the value of y is found in terms of x by means of the equation of the curve: thus, if $y = f(x)$ be this equation, the area is represented by

$$\int f(x) dx$$

taken between suitable limits.

Conversely the value of any definite integral, such as

$$\int_a^b f(x) dx$$

---

[*] This demonstration is substantially that given by Newton; (see Principia, Lib. 1. sect. 1. Lemma 2); and is the geometrical representation of the result established in Art. 86.

The modification in the proof when the elements of BV are considered unequal, but each infinitely small, is easily seen. It may be remarked that the result here given is but a particular case of the general principle laid down in Arts. 38, 39, Dif. Cal.

may be represented geometrically by the area of a definite portion of the curve represented by the equation

$$y = f(x).$$

Again, it is plain that the area between the curve, the axis of y, and two ordinates to that axis, is represented by

$$\int x\,dy,$$

taken between the proper limits: the co-ordinate axes being supposed rectangular.

We proceed to illustrate this method of determining areas by a few applications, commencing with the simplest examples.

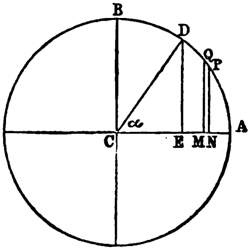
125. **The Circle.**—Taking the equation of a circle in the form

$x^2 + y^2 = a^2$, we get $y = \sqrt{a^2 - x^2}$,

and the area is represented by

$$\int \sqrt{a^2 - x^2}\,dx,$$

taken between proper limits.



For instance, to find the area of the portion represented by *APDE* in the accompanying figure. Let $x = a\cos\theta$, then the area in question is plainly represented by

$$a^2\int_0^{\alpha} \sin^2\theta\,d\theta = \frac{a^2}{2}(\alpha - \sin\alpha\cos\alpha);\ \text{where } \alpha = \angle DCA.$$

This result is also evident from geometry; for, the area *DPAE* is the difference between *DCAP* and *DCE*, or is

$$\frac{a^2\alpha}{2} - \frac{a^2\sin\alpha\cos\alpha}{2}.$$

The area of the quadrant *ACB* is got by making $\alpha = \frac{\pi}{2}$; and accordingly is $\frac{\pi a^2}{4}$: hence, the entire area of the circle is πa^2.

126. **The Ellipse.**—From the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ we get } y = \frac{b}{a}\sqrt{a^2 - x^2},$$

and the element of area is

$$\frac{b}{a}\sqrt{a^2 - x^2}\,dx,$$

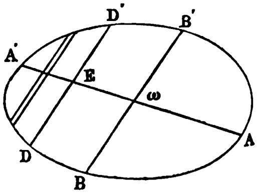
but this is $\frac{b}{a}$ times the area of the corresponding element of the circle whose radius is a: consequently the area of any portion of the ellipse is $\frac{b}{a}$ times that of the corresponding part of the circle. This is also evident from geometry.

The area of the entire ellipse is πab.

Again, if the equation of an ellipse be given in the form $Ax^2 + By^2 = C$, its area is evidently

$$\frac{\pi C}{\sqrt{AB}}.$$

As an application of oblique axes, let it be proposed to find the area of the segment of an ellipse cut off by any chord DD'.



Draw the diameter AA', conjugate to the chord, and BB' parallel to it. Then, C being the centre, let

$$CA' = a', \; CB' = b', \; ACB' = \omega,$$

and the equation of the ellipse is $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$; hence the area $DA'D'$ is represented by

$$2\frac{b'}{a'}\sin\omega\int_{CE}^{CA'}\sqrt{a'^2 - x^2}\,dx = a'b'\sin\omega\,(\alpha - \sin\alpha\cos\alpha),$$

where
$$\cos\alpha = \frac{CE}{CA'}.$$

Again, $a'b'\sin\omega = ab$, by an elementary property of the ellipse, a and b being the semiaxes.

Hence the area of the segment in question is

$$ab(\alpha - \sin\alpha\cos\alpha).$$

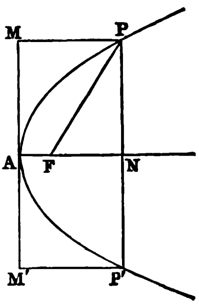
This result can also be deduced immediately from the circle by the method of orthogonal projection.

127. **The Parabola.**—Taking the equation of the parabola in the form

$$y^2 = 4px, \text{ we get } y = 2\sqrt{px}.$$

Hence the area of the portion APN is

$$2p^{\frac{1}{2}}\int x^{\frac{1}{2}}dx, \text{ or } \frac{4}{3}p^{\frac{1}{2}}x^{\frac{3}{2}}, \text{ i.e. } \frac{2}{3}xy.$$



Consequently, the area of the segment PAP' cut off by a chord perpendicular to he axis, is $\frac{2}{3}$ of the rectangle $PMM'P'$.

It is easily seen that a similar relation holds for the segment ut off by any chord.

More generally, let the equation of the curve be $y = ax^n$, where n is positive.

Here $$\int y\,dx = a\int x^n\,dx = \frac{ax^{n+1}}{n+1} + \text{const.}$$

If the area be counted from the origin, the constant vanishes, and the expression for the area becomes

$$\frac{ax^{n+1}}{n+1}, \text{ or } \frac{xy}{n+1}.$$

Hence, the area is in a constant ratio to the rectangle under the co-ordinates. A corresponding result holds for oblique axes. The discussion when n is negative is left to the tudent.

128. **The Hyperbola.**—The simplest form of the equation f an hyperbola is where the asymptotes are taken for cordinate axes: in this case its equation is of the form $xy = c^2$.

Hence, denoting the angle between the asymptotes by ω, the area between the curve and the asymptote is denoted by

$$c^2 \sin\omega \int \frac{dx}{x}, \text{ or } c^2 \sin\omega \log\left(\frac{x_1}{x_0}\right),$$

where x_1 and x_0 are the abscissæ of the limiting points.

If the curve be referred to its axes, its equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

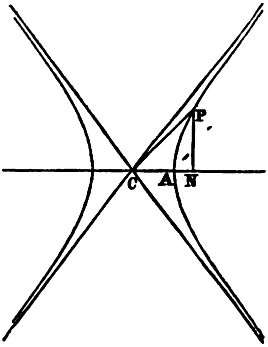
and the element of area ydx, becomes

$$\frac{b}{a} \sqrt{x^2 - a^2}\, dx.$$

Hence the area is represented by

$$\frac{b}{a} \int \sqrt{x^2 - a^2}\, dx,$$

taken between proper limits.



Again, $\int \sqrt{x^2 - a^2}\, dx = \int \frac{x^2 dx}{\sqrt{x^2 - a^2}} - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}.$

Also, integrating by parts, we have

$$\int \sqrt{x^2 - a^2}\, dx = x\sqrt{x^2 - a^2} - \int \frac{x^2 dx}{\sqrt{x^2 - a^2}}.$$

Adding, and dividing by 2, we get

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right).$$

Accordingly, if we suppose the area counted from the ummit A, we have

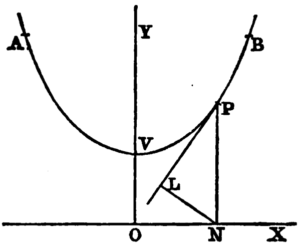
$$APN = \frac{b}{2a}\, x\sqrt{x^2 - a^2} - \frac{ab}{2}\log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right).$$

$$= \frac{xy}{2} - \frac{ab}{2}\log\left(\frac{x}{a} + \frac{y}{b}\right).$$

Again, since the triangle $CPN = \frac{1}{2}\,xy$, it follows that

$$\text{sector } ACP = \frac{ab}{2}\log\left(\frac{x}{a} + \frac{y}{b}\right). \qquad (1)$$

129. **The Catenary.**—If an inelastic string of uniform density be allowed to hang freely from two fixed points, the curve which it assumes is called the catenary.



Its equation is easily shown rom mechanical considerations, vhen reduced to its simplest form, o be

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

The shape of the curve is exibited in the accompanying figure, which the distance from the origin O to the lowest point of the curve is equal to a.

The area of any portion $VPNO$ is

$$\frac{a}{2}\int_0^x \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)dx$$

$$= \frac{a^2}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) = a\,(y^2 - a^2)^{\frac{1}{2}}. \qquad (2)$$

EXAMPLES.

1. In the equilateral hyperbola, $x^2 - y^2 = 1$, if double the sectorial area ACP (figure of Art. 128), be denoted by S, prove that

$$x = \frac{e^S + e^{-S}}{2}, \quad y = \frac{e^S - e^{-S}}{2}.$$

These may be written in another form, viz.:—

$$x = \cos(S\sqrt{-1}), \quad y = \sin(S\sqrt{-1}).$$

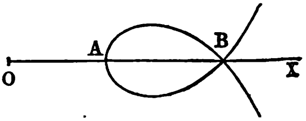
These functions are called the ***hyperbolic sine*** and ***cosine*** of S, and have many analogies to ordinary sines and cosines.

2. To find the area of the oval of the parabola of the third degree, with a double point,

$$cy^2 = (x - a)(x - b)^2.$$

The area in question is represented by

$$\frac{2}{\sqrt{c}}\int_a^b (b - x)\sqrt{x - a}\,dx.$$



Let $x - a = z^2$, and we easily find for the area*

$$\frac{8(b - a)^{\frac{5}{2}}}{3 \cdot 5\, c^{\frac{1}{2}}}.$$

3. Find the whole area of the curve

$$a^2y^2 = x^3(2a - x). \qquad \textit{Ans. } \pi a^2.$$

4. Find the whole area between the cissoid

$$x^3 = y^2(a - x)$$

and its asymptote.

Since $x - a = 0$ is the equation of the asymptote, the area in question is represented by

$$\int_0^a \frac{x^{\frac{3}{2}}\,dx}{(a - x)^{\frac{1}{2}}}.$$

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[*] The student will find little difficulty in proving that this area is $\frac{2\sqrt{2}}{5}$ times the rectangle which circumscribes the oval, having its sides parallel to the co-ordinate axes.

Let $x = a \sin^2\theta$, and this becomes

$$2a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta :$$

hence the area in question is $\frac{3}{8} \pi a^2$.

5. Find the area of the loop of the curve

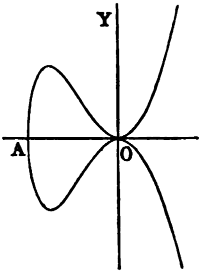
$$a^3 y^2 = x^4 (b + x).$$

This curve has been considered in Art. 255, Dif. Cal. Its form is exhibited in the annexed figure; and the area of the loop is plainly

$$\frac{2}{a^{\frac{3}{2}}} \int_{-b}^{0} x^2 \sqrt{b + x} \, dx.$$

Let $b + x = z^2$, and it is easily seen that the area in question is represented by

$$\frac{8 \cdot b^{\frac{7}{2}}}{3 \cdot 5 \cdot 7 \cdot a^{\frac{3}{2}}}$$

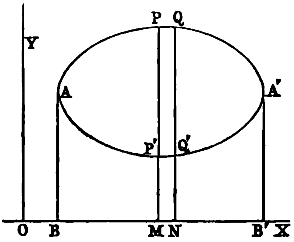


6. Find the area between the witch of Agnesi

$$xy^2 = 4a^2 (2a - x)$$

and its asymptote. *Ans.* $4\pi a^2$.

130. In finding the whole area of a closed curve, such as that represented in the figure, we suppose lines, PM, QN, &c., drawn parallel to the axis of y, and, assuming each of these lines to meet the curve in but two points, and making $PM = y_2$, $P'M = y_1$, the elementary area $PQQ'P'$ is represented by $(y_2 - y_1)\,dx$, and the entire* area by



$$\int_{OB}^{OB'} (y_2 - y_1)\,dx ;$$

in which OB, OB' are the limiting values of x.

For example, let it be proposed to find the whole area of an ellipse given by the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

---

* This form still holds when the axis of x intersects the curve, for the ordinates below that axis have a negative sign, and $(y_2 - y_1)\,dx$ will still represent the element of the area between two parallel ordinates.

Here, solving for y, we easily find

$$y_2 - y_1 = \frac{2}{b}\sqrt{(h^2 - ab)\,x^2 + 2\,(hf - bg)\,x + f^2 - bc}.$$

Also, the limiting values of x are the roots of the quadratic expression under the radical sign.

Accordingly, denoting these roots by α and β, and observing that $h^2 - ab$ is negative for an ellipse, the entire area is represented by

$$\frac{2\sqrt{ab - h^2}}{b}\int_\alpha^\beta \sqrt{(x - \alpha)\,(\beta - x)}\,dx.$$

To find this, assume $x - \alpha = (\beta - \alpha)\sin^2\theta$,

then $\beta - x = (\beta - \alpha)\cos^2\theta$, and we get

$$\int_\alpha^\beta \sqrt{(x - \alpha)\,(\beta - x)}\,dx = 2(\beta - \alpha)^2\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{\pi}{8}(\beta - \alpha)^2.$$

Again $(\beta - \alpha)^2 = 4\cdot\dfrac{(hf - bg)^2 + (f^2 - bc)\,(ab - h^2)}{(ab - h^2)^2}$

$$= \frac{4b\,(af^2 + bg^2 + ch^2 - 2fgh - abc)}{(ab - h^2)^2}.$$

Hence the area of the ellipse is represented by

$$\frac{\pi\,(af^2 + bg^2 + ch^2 - 2fgh - abc)}{(ab - h^2)^{\frac{3}{2}}}.$$

This result can be verified without difficulty, by determining the value of the rectangle under the semiaxes of an ellipse, in terms of the coefficients of its general equation.

It is worthy of observation that if we suppose a closed curve to be described by the motion of a point round its entire perimeter, the whole inclosed area is represented by $\int y\,dx$, taken for every point around the entire curve.

Thus, in the preceding figure, if we proceed from A to A' along the upper portion of the curve, the corresponding part of the integral $\int ydx$ represents the area $APA'B'B$. Again, in returning from A' to A along the lower part of the curve, the increment dx is negative, and the corresponding part of $\int ydx$ is also negative (assuming that the curve does not intersect the axis of x) and represents the area $A'P'ABB'$, taken with a negative sign. Consequently the whole area of the closed curve is represented by the integral $\int ydx$, taken for all points on the curve.

The student will find no difficulty in showing that this proof is general, whatever be the form of the curve, and whatever the number of points in which it is met by the parallel ordinates.

To avoid ambiguity the preceding result may be stated as follows :—*The area of any closed curve is represented by*

$$\int y \frac{dx}{ds} ds$$

taken through the entire perimeter of the curve, the element of the curve being regarded as positive throughout.

131. In many cases, instead of determining y in terms of x, we can express them both in terms of a single variable, and thus determine the area by expressing its element in terms of that variable.

For instance, in the ellipse, if we make $x = a \sin \phi$, we get $y = b \cos \phi$, and ydx becomes $ab \cos^2\phi \, d\phi$, the integral of which gives the same result as before.

In like manner, to find the area of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1;$$

let $x = a \sin^3\phi$, then $y = b \cos^3\phi$, and ydx becomes

$$3ab \sin^2\phi \cos^4\phi \, d\phi :$$

hence, the entire area of the curve is represented by

$$12ab \int_0^{\frac{\pi}{2}} \sin^2\phi \cos^4\phi \, d\phi = \frac{3}{8} \pi ab.$$

EXAMPLES.

1. Find the whole area of the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \textit{Ans.}\ \frac{3\pi(a^2 - b^2)^2}{8ab}.$$

2. Find the whole area of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{2m+1}} + \left(\frac{y}{b}\right)^{\frac{2}{2n+1}} = 1.$$

$$\textit{Ans.}\ \frac{1 \cdot 3 \cdot 5 \ldots (2m+1) \cdot 1 \cdot 3 \cdot 5 \ldots (2n+1)}{2 \cdot 4 \cdot 6 \ \ldots\ldots\ldots\ldots\ 2(m+n+1)} \pi ab.$$

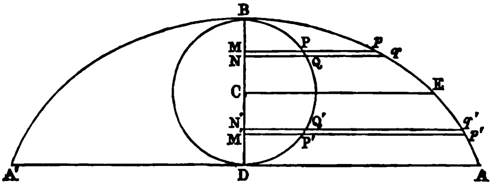
132. **The Cycloid.**—In the cycloid, we have (Dif. Cal., Art. 263),

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta),$$

$$\therefore \int y\,dx = a^2 \int (1 - \cos\theta)^2\, d\theta = 4a^2 \int \sin^4\frac{\theta}{2}\, d\theta.$$

Taking θ between 0 and π, we get $3\pi a^2$ for the entire area between the cycloid and its base.

The area of the cycloid admits also of an elementary geometrical proof, as follows:—



It is obviously sufficient to find the area between the semicircle BPD and the semi-cycloid BpA. To determine this, let points P and P' be taken on the semicircle such that arc BP = arc DP': draw MPp and $M'P'p'$ perpendicular to BD. Take MN and $M'N'$ of equal length, and draw Nq and $N'q'$, also perpendicular to BD: then, by the fundamental property of the cycloid, the line Pp = arc BP, and $P'p'$ = arc BP'

$$\therefore Pp + P'p' = \text{semicircle} = \pi a.$$

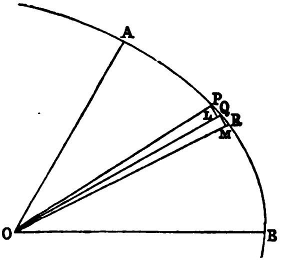
Now, if the interval MN be regarded as indefinitely small, the sum of the elementary areas $PpqQ$ and $P'p'q'Q'$ is equal to the rectangle under MN and the sum of Pp and $P'p'$, or to $\pi a \times MN$.

Again, if the entire figure be supposed divided in like manner, it is obvious that the whole area between the semicircle and the cycloid is equal to πa multiplied by the sum of the elements MN, taken from B to the centre C, i. e. equal to πa^2.

Consequently the whole area of the cycloid is $3\pi a^2$, as before.

The area of a prolate or curtate cycloid can be obtained in like manner.

133. **Areas in Polar Co-ordinates.**—Suppose the curve APB to be referred to polar co-ordinates, O being the pole, and let OP, OQ, OR represent consecutive radii vectores, and PL, QM, arcs of circles described with O as centre. Then the area $OPQ = OPL + PLQ$; but PLQ becomes evanescent in comparison with OPL when P and Q are infinitely near points; consequently the elementary area OPQ = area OPL (in the limit) $= \frac{r^2 d\theta}{2}$; r and θ being the polar co-ordinates of P.



Hence the sectorial area AOB is represented by

$$\tfrac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta,$$

where α and β are the values of θ corresponding to the limiting points A and B.

For example, let it be proposed to find the area of the locus of the foot of the perpendicular from the centre on a tangent to an ellipse.

Writing the equation of the ellipse in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of the locus in question is obviously

$$r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta:$$

Hence its area is

$$\frac{a^2}{2}\int \cos^2\theta\,d\theta + \frac{b^2}{2}\int \sin^2\theta\,d\theta = \frac{a^2+b^2}{4}\,\theta + \frac{a^2-b^2}{4}\sin\theta\cos\theta.$$

The entire area of the locus is

$$\frac{\pi}{2}(a^2+b^2).$$

The equation of the corresponding locus for the hyperbola is

$$r^2 = a^2\cos^2\theta - b^2\sin^2\theta.$$

In finding its area, since r must be real, we must have $a^2\cos^2\theta - b^2\sin^2\theta$ positive : accordingly, the limits for θ are 0 and $\tan^{-1}\frac{a}{b}$.

Integrating between these limits, and multiplying by four, we get for the entire area

$$ab + (a^2-b^2)\tan^{-1}\frac{a}{b}.$$

In this case, if we had at once integrated between $\theta = 0$ and $\theta = 2\pi$, we should have found for the area $(a^2-b^2)\frac{\pi}{2}$. This anomaly would arise from our having integrated through an interval for which r^2 is negative, and for which, therefore, the corresponding part of the curve is imaginary.

Examples.

1. Show that the entire area of the Lemniscate

$$r^2 = a^2\cos 2\theta$$

is a^2.

2. In the hyperbolic spiral

$$r\theta = a,$$

prove that the area bounded by any two radii vectores is proportional to the difference between their lengths.

3. Find the area of the loop of the Folium of Descartes, whose equation is

$$x^3 + y^3 = 3axy.$$

Transforming to polar co-ordinates we have

$$r = \frac{3a \cos\theta \sin\theta}{\sin^3\theta + \cos^3\theta}.$$

Again, the limiting values of θ are 0 and $\frac{\pi}{2}$,

$$\therefore \text{Area} = \frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cos^2\theta \, d\theta}{(\sin^3\theta + \cos^3\theta)^2}.$$

Let $\tan\theta = u$, and this expression becomes

$$\frac{9a^2}{2}\int_0^{\infty} \frac{u^2 du}{(1 + u^3)^2} = \frac{3a^2}{2}.$$

4. To find the area of the Limaçon

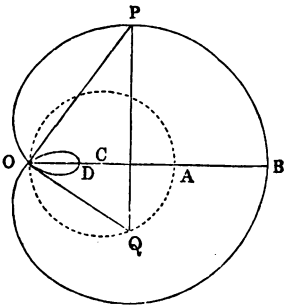
$$r = a \cos\theta + b.$$

Here we must distinguish between two cases.

(1). Let $b > a$. In this case the curve consists of one loop, and its area is

$$\tfrac{1}{2}\int_0^{2\pi} (a \cos\theta + b)^2 d\theta = \left(b^2 + \frac{a^2}{2}\right)\pi.$$

When $b = a$, the curve becomes a Cardioid, and the area $\frac{3\pi a^2}{2}$.



(2). Let $b < a$. The curve in this case has two loops, as in the figure (see Dif. Cal., Art. 273), the outer loop corresponding to $r = a \cos\theta + b$, the inner to

$$r = a \cos\theta - b.$$

To find the area of the inner loop, we take θ between the limits 0 and α, where $\alpha = \cos^{-1}\left(\frac{b}{a}\right)$; and the entire area is

$$\int_0^{\alpha} (a \cos\theta - b)^2 \, d\theta$$

$$= \int_0^{\alpha} (a^2 \cos^2\theta - 2ab \cos\theta + b^2) \, d\theta$$

$$= \left(\frac{a^2}{2} + b^2\right)\alpha + \frac{a^2}{2} \sin\alpha \cos\alpha - 2ab \sin\alpha$$

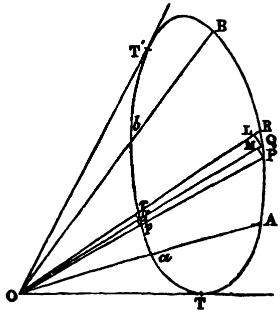
$$= \left(\frac{a^2}{2} + b^2\right)\cos^{-1}\frac{b}{a} - \frac{3}{2} b\sqrt{a^2 - b^2}.$$

It is easily seen that the sum of the areas of the two loops is obtained by integrating between the limits 0 and 2π, and accordingly is

$$\pi\left(\frac{a^2}{2} + b^2\right),$$

as in the former case.

134. In finding the whole area of a closed curve by polar co-ordinates we distinguish between two cases. When the origin O is outside, we suppose tangents OT, OT', drawn from O, and vectors OP, OQ, &c., drawn to cut the curve; then, if these lines intersect it in but two points each, the element of area $PpqQ$ is the difference between the areas POQ and pOq; or in the limit, is $\frac{1}{2}(r_1^2 - r_2^2)\,d\theta$, where $OP = r_1$, $Op = r_2$.



Hence, the expression

$$\tfrac{1}{2}\int(r_1^2 - r_2^2)\,d\theta,$$

taken between the limits corresponding to the tangents OT and OT', represents the entire included area.

If the origin lie inside the curve, its whole area is in general represented by $\frac{1}{2}\int(r_1^2 + r_2^2)\,d\theta$, taken between the limits $\theta = 0$, and $\theta = \pi$.

We shall illustrate these results by applying them to the circle

$$r^2 - 2rc\cos\theta + c^2 = a^2.$$

If the origin be outside, we have $c > a$, and $r_1 + r_2 = 2c\cos\theta$, and $r_1 r_2 = c^2 - a^2$, $\therefore\ r_1 - r_2 = 2\sqrt{a^2 - c^2\sin^2\theta}$.

Hence $(r_1^2 - r_2^2)\,d\theta = 4c\cos\theta\sqrt{a^2 - c^2\sin^2\theta}\,d\theta$; and the limiting values of θ are $\pm\sin^{-1}\dfrac{a}{c}$.

Hence the whole area is

$$2c\int_{-\sin^{-1}\frac{a}{c}}^{\sin^{-1}\frac{a}{c}}\cos\theta\sqrt{a^2 - c^2\sin^2\theta}\,d\theta.$$

Let $c\sin\theta = a\sin\phi$, and this integral transforms into

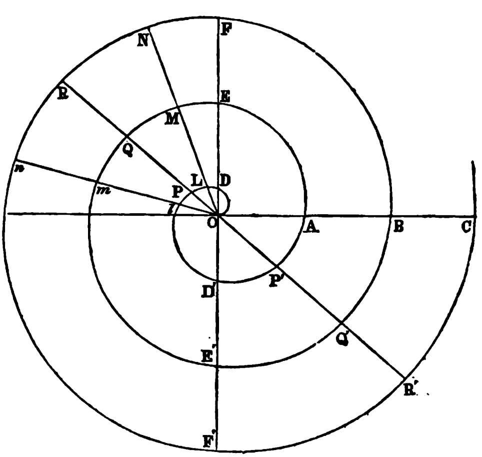
$$2a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\phi\,d\phi = \pi a^2.$$

Again, if the origin be inside, we have $c < a$, and

$$\tfrac{1}{2}(r_1^2 + r_2^2) = a^2 + c^2 \cos 2\theta,$$

$$\therefore \tfrac{1}{2}\int_0^\pi (r_1^2 + r_2^2)\, d\theta = \int_0^\pi (a^2 + c^2 \cos 2\theta)\, d\theta = \pi a^2.$$

The method given above may be applied to find the area included between two branches of the same spiral curve. As an example, let us consider the case of the spiral of Archimedes.



135. **The Spiral of Archimedes.**—The equation of this curve is $r = a\theta$, and its form, for positive values of θ, is represented in the accompanying figure, in which O is the pole and OA the line from which θ is measured. Let any line drawn through O meet the different branches of the spiral in points P, Q, R, &c., then, if $OP = r$, and $POA = \theta$, we have, from the equation of the curve,

$$OP = a\theta, \quad OQ = a(\theta + 2\pi), \quad OR = a(\theta + 4\pi), \text{ \&c.}$$

Hence, $PQ = QR =$ &c., $= 2a\pi = c$ (suppose); i. e. the intercepts between any two consecutive branches of the spiral are of constant length.

Again, let $OQ = r_1$, $OR = r_2 = r_1 + c$, and the area between the two corresponding branches is

$$\tfrac{1}{2}\int (r_2^2 - r_1^2)\, d\theta = c\int r_1\, d\theta + \frac{c^2}{2}\int d\theta.$$

Now, suppose MN and mn represent the limiting lines, and let β and α be the corresponding values of θ; then the area $nNMm$ will be equal to

$$c\int_{\alpha}^{\beta} a\theta\, d\theta + \frac{c^2}{2}\int_{\alpha}^{\beta} d\theta = \frac{c}{2}(\beta - \alpha)(a\alpha + a\beta + c)$$

$$= \frac{c}{2}(\beta - \alpha)(OM + On). \qquad (3)$$

If $\beta - \alpha = \pi$, this gives for the area of the portion between two consecutive branches $QE'Q'$ and $RF'R'$, intercepted by any right line RR' drawn through the pole, $\frac{\pi}{2} RQ.QR'$, i. e. half the area of the ellipse whose semi-axes are RQ and $R'Q$.

136. The formula in Article 134 still holds obviously when AB and ab represent portions of different curves.

It is also easily seen, as in (130), that if a point be supposed to move round any closed boundary, the included area is in all cases represented by $\frac{1}{2}\int r^2 d\theta$, taken round the entire boundary, whatever be its form; the elementary angle $d\theta$ being taken with its proper sign throughout.

Again, if we transform to rectangular axes by the relations $x = r\cos\theta$, $y = r\sin\theta$, we get

$$\tan\theta = \frac{y}{x}, \therefore \frac{d\theta}{\cos^2\theta} = \frac{xdy - ydx}{x^2}.$$

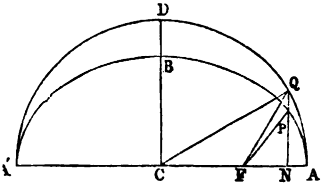
Hence $$r^2\, d\theta = xdy - ydx;$$

and the area swept out by the radius vector is represented by the integral

$$\tfrac{1}{2}\int(xdy - ydx),$$

taken between suitable limits; a result which can also be easily arrived at geometrically.

137. **Area of Elliptic Sector. Lambert's Theorem.**—It is of importance in Astronomy to be able to express the area AFP swept out by the focal radius vector of an ellipse. This can be arrived at by integration from the polar equation of the curve; it is, however, more easily obtained geometrically.



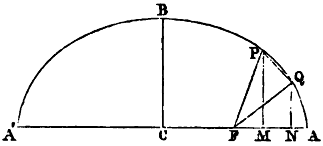
For, if the ordinate PN be produced to meet the auxiliary circle in Q, we have

$$\text{area } AFP = \frac{b}{a} \times \text{area } AFQ = \frac{b}{a}(ACQ - CFQ)$$

$$= \frac{ab}{2}(u - e \sin u), \qquad (4)$$

where $u = \angle ACQ$.

By aid of this result, the area of any elliptic sector can be expressed in terms of the focal distances of its extremities, and of the chord joining them.



For, let QFP represent the sector, and let $FP = \rho$, $FQ = \rho'$, $PQ = c$; then, denoting by u and u' the eccentric angles corresponding to P and Q, the area of the sector QFP, by (4), is represented by

$$\frac{ab}{2}\left\{u - u' - e(\sin u - \sin u')\right\}.$$

We proceed to show that this result can be written in the form

$$\frac{ab}{2}\{\phi - \phi' - (\sin\phi - \sin\phi')\}, \qquad (5)$$

where ϕ and ϕ' are given by the equations

$$\sin\frac{\phi}{2} = \tfrac{1}{2}\sqrt{\frac{\rho + \rho' + c}{a}}, \quad \sin\frac{\phi'}{2} = \tfrac{1}{2}\sqrt{\frac{\rho + \rho' - c}{a}}.$$

For, assume that ϕ and ϕ' are determined by the equations

$$u - u' = \phi - \phi', \; e(\sin u - \sin u') = \sin\phi - \sin\phi'. \qquad (a)$$

The latter gives

$$e \sin\frac{u - u'}{2}\cos\frac{u + u'}{2} = \sin\frac{\phi - \phi'}{2}\cos\frac{\phi + \phi'}{2},$$

or

$$e \cos\frac{u + u'}{2} = \cos\frac{\phi + \phi'}{2}.$$

Again, since the co-ordinates of P and Q are $a \cos u$, $b \sin u$, and $a \cos u'$, $b \sin u'$, respectively, we have

$$c^2 = a^2 (\cos u - \cos u')^2 + b^2 (\sin u - \sin u')^2$$

$$= 4 \sin^2 \frac{u - u'}{2} \left(a^2 \sin^2 \frac{u + u'}{2} + b^2 \cos^2 \frac{u + u'}{2} \right)$$

$$= 4a^2 \sin^2 \frac{u - u'}{2} \left(1 - e^2 \cos^2 \frac{u + u'}{2} \right)$$

$$= 4a^2 \sin^2 \frac{\phi - \phi'}{2} \sin^2 \frac{\phi + \phi'}{2};$$

$$\therefore c = 2a \sin \frac{\phi - \phi'}{2} \sin \frac{\phi + \phi'}{2} = a (\cos \phi' - \cos \phi). \qquad (b)$$

Again, from the ellipse, we have

$$\rho = a (1 - e \cos u), \quad \rho' = a (1 - e \cos u'),$$

$$\therefore \rho + \rho' = 2a - ae (\cos u + \cos u') = 2a - 2ae \cos \frac{u + u'}{2} \cos \frac{u - u'}{2}$$

$$= 2a - 2a \cos \frac{\phi + \phi'}{2} \cos \frac{\phi - \phi'}{2} = 2a - a (\cos \phi + \cos \phi'). \qquad (c)$$

Hence, adding and subtracting (b) and (c), we get

$$\frac{\rho + \rho' + c}{a} = 2 (1 - \cos \phi) = 4 \sin^2 \frac{\phi}{2}$$

$$\frac{\rho + \rho' - c}{a} = 2 (1 - \cos \phi') = 4 \sin^2 \frac{\phi'}{2},$$

which proves the theorem in question.

Consequently, the *area** *of any focal sector of an ellipse can be expressed in terms of the focal distances of its extremities, of the chord which joins them, and of the axes of the curve.*

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* This remarkable result is an extension, by Lambert, of the corresponding formula for a parabola given by Euler in Miscell. Berolin. T. vii., p. 20. It furnishes an expression for the time of describing any arc of a planet's orbit, in terms of its chord, the distances of its extremities from the sun, and the major axis of the orbit; neglecting the disturbing action of the other bodies of the solar system.

138. We next proceed to an elementary principle which is sometimes useful in determining areas, viz. :—

The area of any portion of the curve represented by the equation

$$F\left(\frac{x}{a}, \frac{y}{b}\right) = c$$

is ab times the area of the corresponding portion of the curve

$$F(x, y) = c.$$

This result is obvious, for the former equation is transformed into the latter, by the assumption $\frac{x}{a} = x'$, $\frac{y}{b} = y'$; and hence, ydx becomes $aby'dx'$;

$$\therefore \int ydx = ab\int y'dx',$$

the integrals being taken through corresponding limits ; a result which is also easily shown by projection.

Thus, for example, the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ reduces to that of the circle ; and the area of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

to that of the equilateral hyperbola $x^2 - y^2 = 1$.

Again, let it be proposed to find the area of the curve

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{l^2} + \frac{y^2}{m^2}.$$

The transformed equation is

$$(x^2 + y^2)^2 = \frac{a^2x^2}{l^2} + \frac{b^2y^2}{m^2},$$

or, in polar co-ordinates,

$$r^2 = \frac{a^2\cos^2\theta}{l^2} + \frac{b^2\sin^2\theta}{m^2}.$$

But the whole area of this (Art. 133) is $\frac{\pi}{2}\left(\frac{a^2}{l^2} + \frac{b^2}{m^2}\right)$.

Consequently the whole area of the proposed is

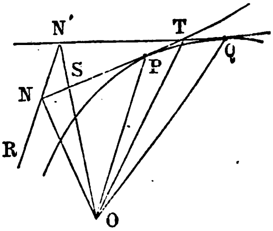
$$\frac{\pi}{2}\, ab \left(\frac{a^2}{l^2} + \frac{b^2}{m^2}\right).$$

It may be remarked that the equations

$$F\left(\frac{x}{a}, \frac{y}{a}\right) = c, \quad F(x, y) = c,$$

represent similar curves; and their corresponding linear dimensions are as $a : 1$; consequently the areas of similar curves are as the squares of their dimensions.

139. **Area of a Pedal Curve.**—If from any point perpendiculars be drawn to the tangents to any curve, the locus of their feet is a new curve, called the pedal of the original (Dif. Cal. Art. 183).



If p and ω be the polar co-ordinates of N the foot of the perpendicular from the origin O, then the polar element of area of the locus described by N is plainly $\frac{p^2 d\omega}{2}$, and the sectorial area of any portion is accordingly

$$\tfrac{1}{2} \int p^2 d\omega$$

taken between proper limits.

There is another expression for the area of a closed pedal curve which is sometimes useful.

Let S_1 denote the whole area of the pedal, and S that of the original curve; then the area included between the two curves is ultimately equal to the sum of the elements represented by NTN' in the figure.

Hence

$$S_1 = S + \Sigma NTN' = S + \tfrac{1}{2} \int PN^2\, d\omega. \qquad (6)$$

Again, by the preceding,

$$S_1 = \tfrac{1}{2} \int ON^2\, d\omega.$$

Accordingly, by addition,

$$2S_1 = S + \tfrac{1}{2} \int OP^2\, d\omega. \qquad (7)$$

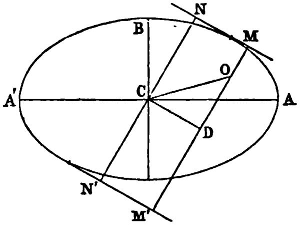
It is easily seen that equation (6) admits of being stated herwise, viz.:—

The whole area of the pedal of any closed curve is equal to e sum of the areas of the curve and of the pedal of its evolute: th pedals having the same origin.

For, PN is equal in length to the perpendicular from O the normal at P: and hence $\frac{1}{2}PN^2 d\omega$ represents the ele- ent of area of the locus described by the foot of this perpen- cular, i. e. of the pedal of the evolute of the original curve.

For example, it follows from Art. 133 that the area of e pedal of the evolute of an ellipse is $\frac{\pi}{2}(a-b)^2$, the centre eing origin.

140. **Area of Pedal of Ellipse for any Origin.**—Suppose to be the pedal origin, and M, OM' perpendiculars on wo parallel tangents to the llipse, draw CN the perpendi- ular from C, let $OM = p_1$, $M' = p_2$, $CN = p$, $OC = c$, $OCA = \alpha$, $\angle ACN = \omega$; then



$$p_1 = MD - OD = p - c\cos(\omega - \alpha),$$

$$p_2 = p + c\cos(\omega - \alpha).$$

Again, the whole area of the pedal is

$$\tfrac{1}{2}\int_0^\pi (p_1^2 + p_2^2)\,d\omega = \int_0^\pi \{p^2 + c^2\cos^2(\omega - \alpha)\}\,d\omega$$

$$= \int_0^\pi p^2 d\omega + c^2\int_0^\pi \cos^2(\omega - \alpha)\,d\omega = \frac{\pi}{2}(a^2 + b^2 + c^2). \qquad (8)$$

That is, the area of the pedal with respect to O as origin ceeds the area of its pedal with respect to C by half the ea of the circle whose radius is OC.

If the origin O lie outside the ellipse, the pedal consists two loops intersecting at O and lying one inside the her; and in that case the expression in (8) represents the m of the areas of the two loops, as can be easily seen.

The result established above is a particnlar case of a general theorem of Steiner, which we next proceed to consider.

141. Suppose A to be the whole area of the pedal of any closed curve with respect to any internal origin O, and A' the area with respect to another origin O'; then, if p and p' be the lengths of the perpendiculars from O and O' on a tangent to the curve, we have

$$A = \frac{1}{2}\int_0^{2\pi} p^2 d\omega, \quad A' = \frac{1}{2}\int_0^{2\pi} p'^2 d\omega.$$

Also, adopting the notation of the last article,

$$p' = p - c\cos(\omega - a) = p - x\cos\omega - y\sin\omega;$$

where x, y represent the co-ordinates of O' with respect to rectangular axes drawn through O. Hence we get

$$A' - A = \frac{1}{2}\int_0^{2\pi} (x\cos\omega + y\sin\omega)^2 d\omega$$

$$- x\int_0^{2\pi} p\cos\omega\, d\omega - y\int_0^{2\pi} p\sin\omega\, d\omega.$$

But $\int_0^{2\pi} \cos^2\omega\, d\omega = \pi$, $\int_0^{2\pi} \sin^2\omega\, d\omega = \pi$, $\int_0^{2\pi} \sin\omega\cos\omega\, d\omega = 0$.

Also, for a given curve, $\int_0^{2\pi} p\cos\omega\, d\omega$ and $\int_0^{2\pi} p\sin\omega\, d\omega$ are constants when O is given. Denoting their values by g and h, we have

$$A' - A = \frac{\pi}{2}(x^2 + y^2) - gx - hy. \qquad (9)$$

This equation shows that if O be fixed, the locus of the origin O', for which the area of the pedal of a closed area is constant, is a circle.* The centre of this circle is the same hatever be the given area, and all the circles got by varying ıe pedal area are concentric.

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* It can be seen, without difficulty, from the demonstration given above, that when the curve is not closed the locus of the origin for pedals of equal area is an ellipse.

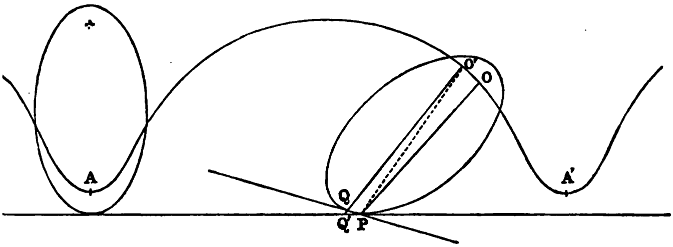
If the origin O be supposed taken at the centre of this ırcle, the constants g and h will disappear, and, in this case, ıe pedal area is a minimum, and the difference between the reas of the pedals is equal to half the area of the circle whose ıdius is the distance between the pedal origins.

For example, if we take the origin at the centre, the edal of a circle is the circle itself. For any other origin the edal is a limaçon; hence the whole area of a limaçon is $\cdot\left(a^2 + \frac{b^2}{2}\right)$, as found in Art. 133, Ex. 4.

142. **Areas of Roulettes.**—The connexion between the reas of roulettes and of pedals is contained in a very elegant heorem,* also due to Steiner, which may be stated as folɔws:—

Where a closed curve rolls on a right line, the area beween the right line and the roulette generated in a complete evolution by any point invariably connected with the rolling urve is double the area of the pedal of the rolling curve, his pedal being taken with respect to the generating point s origin.

To prove this, suppose O to be the describing point in any



osition of the rolling curve, and P the corresponding point f contact, let O' represent an infinitely near position of the describing point, Q' the corresponding point of contact, and Q a point on the curve such that $PQ = PQ'$; then Q is the point which coincides with Q' in the new position of the rolling curve; and, denoting the angle between the tangents at P and Q (the angle of contingence) by $d\omega$, we have $OPO' = d\omega$, since we may regard the curve as turning round P at the instant (Dif. Cal., Art. 268).

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* See Crelle's Journal, Vol. xxi. The corresponding theorem of Steiner nnecting the lengths of roulettes and pedals will be given in the next chapter.

By the area of a roulette we understand the area between the roulette, e fixed line, and the normals drawn at the extremities of the roulette.

Moreover, QQ' ultimately is infinitely small in comparison with QP, and consequently the elementary area $OPQ'O'$ is ultimately the sum of the areas POO' and $QO'P$, neglecting an area which is infinitely small in comparison with either of these areas.

Again, if $OP = r$, we have $POO' = \frac{r^2 d\omega}{2}$, and area $QO'P = QOP$ in the limit.

Also the sum of the elements QOP in an entire revolution is equal to the area (S) of the rolling curve. Consequently, the entire area of the roulette described by O is

$$S + \tfrac{1}{2}\int r^2 d\omega.$$

But we have already seen (7) that this is double the area of the pedal of the curve with respect to the point O; which establishes our proposition.

Again, from Art. 141, it follows that there is one point in any closed curve for which the entire area of the corresponding roulette is a minimum. Also, the area of the roulette described by any other point exceeds that of the minimum roulette by the area of the circle whose radius is the distance between the points.

For instance, if a circle roll on a right line, its centre describes a parallel line, and the area between these lines after a complete revolution is equal to the rectangle under the radius of the circle and its circumference; i. e. is $2\pi a^2$, denoting the radius by a.

Consequently, for a point on the circumference, the area generated is $2\pi a^2 + \pi a^2$, or $3\pi a^2$; which agrees with the area found already for the cycloid.

In like manner, by Steiner's theorem, the area of the ordinary cycloid is the same as that of a cardioid; and the area of a prolate or curtate cycloid the same as that of a limaçon.

Again, if an ellipse roll on a right line, the area of the path described by any point can be immediately obtained.

For example, the pedal of an ellipse with respect to a focus is the circle described on its axis major. Hence, if an ellipse roll upon a right line, the area of the roulette described by its focus in a complete revolution is double the area of the auxiliary circle. Also the area of the roulette described by the centre of the ellipse is equal to the sum of the circles described on the axes of the ellipse as diameters, and is less than the area of the roulette described by any other point.

143. If the curve, instead of rolling on a right line, roll on another curve, it is easily seen that the method of proof given in the last article still holds; provided we take, instead of $d\omega$, the sum of the angles of contingence of the two curves at the point P.

Hence the element of area OPP' is in this case

$$\tfrac{1}{2}\, OP^2\, d\omega \left(1 + \frac{d\omega'}{d\omega}\right), \text{ or } \tfrac{1}{2}\, OP^2\, d\omega\left(1 + \frac{\rho}{\rho'}\right),$$

where ρ and ρ' are the radii of curvature at P, of the rolling and fixed curves, respectively.

Hence, it follows that the area between the roulette, the fixed curve, and the two extreme normals, after a complete revolution is represented by

$$S + \tfrac{1}{2}\int r^2 d\omega\left(1 + \frac{\rho}{\rho'}\right).$$

If a closed curve roll on a curve identical with itself, having corresponding points always in contact, the formula for the area generated becomes

$$S + \int r^2 d\omega.$$

In this case, the area generated is four times that of the corresponding pedal; a result which can also be shown immediately geometrically by drawing a figure.

EXAMPLES.

1. If A be the area of a loop of the curve $r^m = a^m \cos m\theta$, and A_1 the area of its pedal with respect to the polar origin, prove that

$$A_1 = \left(1 + \frac{m}{2}\right) A.$$

For, it is easily seen that the angle between the radius vector and the perpendicular on the tangent is $m\theta$; and $\therefore \omega = (m + 1)\theta$.

Hence, by Art. 139,

$$2A_1 = A + \frac{m+1}{2} \int r^2 d\theta = (m + 2) A.$$

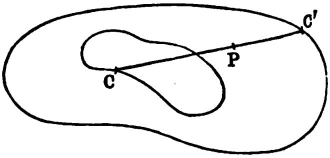
2. If a circle of radius b roll on a circle of radius a, and if A denote the area, after a complete revolution, between the fixed circle, the roulette described by any point, and the extreme normals; and if A' be the area of the pedal of the circle with respect to the generating point, prove that

$$Aa + Bb = 2(a + b) A',$$

where B is the area of the rolling circle.

3. Apply this result to find the area included between the fixed circle and the arc of an epicycloid extending from one cusp to the next.

144. **Theorem.***—If a line CC' of a given length move with its extremities on two fixed closed curves, to find, in terms of the areas of the two fixed curves, an expression for the whole area of the curve generated, in a complete revolution, by any given point P situated on the moving line.



Let $CP = c$, $PC' = c'$, and suppose (x_1, y_1), (x, y), and (x_2, y_2) to be the co-ordinates of the points C, P, and C', respectively, with reference to any rectangular axes.

Then, if θ be the angle made by CC' with the axis of y, we have evidently

$$x_1 = x - c \sin\theta, \quad y_1 = y - c\cos\theta,$$

$$x_2 = x + c' \sin\theta, \quad y_2 = y + c' \cos\theta.$$

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* This simple and elegant theorem appeared, in a modified form, as the Prize Question, by Mr. Holditch, under the name of "Petrarch," in the Lady's and Gentleman's Diary for the year 1858.

Hence we have

$$y_1dx_1 = ydx - c\cos\theta\,(dx + yd\theta) + c^2\cos^2\theta d\theta$$

$$y_2dx_2 = ydx + c'\cos\theta\,(dx + yd\theta) + c'^2\cos^2\theta d\theta.$$

Multiplying the former equation by c', and the latter by c, and adding, we get

$$c'y_1dx_1 + cy_2dx_2 = (c + c')\,ydx + (c + c')\,cc'\cos^2\theta d\theta.$$

$$\therefore\ c'\int y_1dx_1 + c\int y_2dx_2 = (c + c')\int ydx + (c + c')\,cc'\int\cos^2\theta d\theta.$$

If we suppose the rod to make a complete revolution, so as to return to its original position, and if we denote by (C), (C'), (P), the areas of the curves described by the points, C, C', and P, respectively, we shall have (since in this case the angle θ revolves through 2π)

$$c'(C) + c(C') = (c + c')(P) + \pi(c + c')cc'$$

or

$$\frac{c'(C) + c(C')}{c + c'} = (P) + \pi cc'.$$

This determines the area (P) in terms of the areas (C), $C')$ and of the segments c, c'.

When the extremities C, C' move on the same* identical urve we have $(C) = (C')$, and hence $(C) - (P) = \pi cc'$.

Consequently, if *a chord of given length move inside any losed curve, having a tracing point P at the distances c and ′ from its ends, the area comprised between the two curves is qual to* $\pi cc'$.

Should the extremities, instead of revolving, oscillate ack to their former positions, then $(C) = 0$, $(C') = 0$, and . $(P) = -\pi cc'$. The negative sign implies that the area is escribed in a direction contrary to that in which the rod reolves. Should the rod also oscillate, we have $(P) = 0$, which dicates that the area described consists of two equal loops, ne positive and the other negative.

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[*] This holds also where the extremities move on curves of equal area.

145. **Areas by Approximation.**—In many cases it is necessary to approximate to the value of the area included within a closed contour. The usual method is by drawing a convenient number of parallel ordinates at equal intervals; then, when a rough approximation is sufficient, we may regard the area of the curve as that of the polygon got by joining the points of section of the parallel ordinates. Hence, if h be the common distance between the ordinates, and if

$$y_0, y_1, y_2, \&c., y_n,$$

represent the system of parallel ordinates, the area of the polygon, since it consists of a number of trapeziums of equal breadth, is plainly represented by

$$h\left\{\frac{y_0 + y_n}{2} + y_1 + y_2 + \&c. + y_{n-1}\right\}.$$

Hence the rule; add together the halves of the extreme ordinates, and the whole of the intermediate ordinates, and multiply the result by the common interval.

When a nearer approximation is required, the method next in simplicity supposes the curve to consist of a number of parabolic arcs; each parabola having its axis parallel to the equidistant ordinates, and being determined by three of those ordinates.

To find the area of the parabola passing through the points whose ordinates are y_0, y_1, y_2; let $y = \alpha + \beta x + \gamma x^2$ be the equation of the parabola, and, for simplicity, assume the origin at the foot of the intermediate ordinate y_1, then we have

$$y_0 = \alpha - \beta h + \gamma h^2, \quad y_1 = \alpha, \quad y_2 = \alpha + \beta h + \gamma h^2.$$

Again, the area between the first and third ordinate is

$$\int_{-h}^{h} (\alpha + \beta x + \gamma x^2)\, dx = 2h\left(\alpha + \gamma\frac{h^2}{3}\right).$$

But $y_0 + y_2 = 2y_1 + 2\gamma h^2$, hence the area in question is

$$\frac{h}{3}\left\{y_0 + 4y_1 + y_2\right\}.$$

Now, if we suppose the number of intervals n to be even, and add the different parabolic areas, we get, as an approximation to the area, the expression

$$\frac{h}{3}\{y_0 + y_n + 4(y_1 + y_3 + \text{\&c.} + y_{n-1}) + 2(y_2 + y_4 + \text{\&c.} + y_{n-2})\}.$$

Hence the rule. Add together the first and last ordinates, *twice* every second intermediate ordinate, and *four times* each remaining ordinate; and multiply by one-third of the common interval.

We get a closer approximation, by supposing the number of equal intervals a multiple of 3, and regarding the curve as a series of parabolæ of the third degree, each being determined by four equidistant ordinates. To find the area corresponding to one of these parabolic curves, let y_0, y_1, y_2, y_3 be four equidistant ordinates, and for convenience assume the origin midway between y_1 and y_2; then if the equation of the parabolic curve be

$$y = a + \beta x + \gamma x^2 + \delta x^3,$$

and the common interval on the axis of x be denoted by $2h$, we have

$$\begin{aligned} y_0 &= a - 3\beta h + 9\gamma h^2 - 27\delta h^3 \\ y_1 &= a - \beta h + \gamma h^2 - \delta h^3 \\ y_2 &= a + \beta h + \gamma h^2 + \delta h^3 \\ y_3 &= a + 3\beta h + 9\gamma h^2 + 27\delta h^3. \end{aligned}$$

Hence

$$y_0 + y_3 = 2(a + 9\gamma h^2), \quad y_1 + y_2 = 2(a + \gamma h^2).$$

Again, the parabolic area between y_0 and y_3 is

$$\int_{-3h}^{3h} (a + \beta x + \gamma x^2 + \delta x^3)\,dx = 3h(2a + 6\gamma h^2).$$

Substituting in this the values of a and γ obtained from e two preceding equations, the expression for the area comes

$$\frac{3h}{4}\{y_0 + y_3 + 3(y_1 + y_2)\}.$$

If the corresponding expressions be added together, we easily arrive at the following rule.* Add together the first and last ordinates, *twice* every third intermediate ordinate, and *thrice* each remaining ordinate; and multiply by $\frac{3}{8}$ths of the common interval.

It is readily seen that these rules also apply to the approximation to any closed area, by drawing a system of lines, parallel and equidistant, and adopting the intercepts made by the curve instead of the ordinates, in each rule.

Since every definite integral may be represented by a curvilinear area, the methods given above are applicable to the approximate determination of any such integral.

In practice the accuracy of these methods is increased by increasing the number of intervals.

146. **Planimeters.**—Several mechanical contrivances have been introduced for the purpose of practically estimating the area inclosed within any curved boundary. Such instruments are called Planimeters. The simplest and most elegant is that of Professor Amstler of Schaffhausen. It consists of two arms jointed together so as to move in perfect freedom in one plane. A point at the extremity of one arm is made a fixed centre round which the instrument turns; and a wheel is fixed to, and turns on the other arm as an axis, and records by its revolution the area of the figure traced out by a point on this arm. From its construction it is plain that the revolving wheel registers only the motion which is perpendicular to the moving arm on which it revolves.

In the practical application of the instrument it is necessary that the two arms, CA and AB, should return to their original position after the tracing point B has been moved round the entire boundary of the required area.

We shall commence by showing that the length registered by the wheel in a complete revolution is independent of its position on the moving arm; i. e. is the same as if the wheel be supposed placed at the joint.

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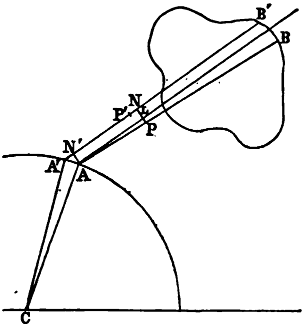
* This and the preceding are commonly called "Simpson's rules" for calculating areas: they were however previously noticed by Newton (see Opuscula. Method. Diff., Prop. 6, scholium) as a particular application of the method of interpolation. The student is referred to Bertrand's Cal. Int. ch. xii., for the more general and accurate methods of approximation of Cotes and Gauss.

To prove this, suppose P to represent the point on the arm at which the centre of the revolving wheel is situated. Let $A'B'$ represent a new position of AB very near to AB, and P' the corresponding position of the point P. Draw PN perpendicular to $A'B'$; then PN is the length registered by the wheel while the arm moves from AB to the infinitely near position $A'B'$.



Next, draw AN' perpendicular, and AL parallel, to $A'B'$.

Let $PN = ds'$, $AN' = ds$, $AP = c$, $PAL = d\phi$, then $PN = PL + AN'$, or $ds' = ds + c\,d\phi$.

Now, if we suppose AB after a complete circuit of the curve, to return to its original position, we have obviously $\Sigma\,(d\phi) = 0$; and therefore $\Sigma\,(ds') = \Sigma\,(ds)$, i. e. the whole length registered by the revolving wheel at P is the same as if it were placed at A.

Next, let x and y be the co-ordinates of B with respect to rectangular axes drawn through C, and let $AC = a$, $AB = b$, $ACX = \theta$; and suppose ϕ the angle which BA produced makes with the axis of x; then we shall have

$$x = a\cos\theta + b\cos\phi,\quad y = a\sin\theta + b\sin\phi.$$

Hence $$xdy - ydx = a^2d\theta + b^2d\phi + ab\cos(\theta - \phi)\,d(\theta + \phi).$$

Also $$ds = AN' = AA'\sin AA'N = a\,d\theta\cos(\theta - \phi).$$

But $$\theta + \phi = 2\theta - (\theta - \phi),$$

$$\therefore ab\cos(\theta - \phi)\,d(\theta + \phi)$$
$$= 2ab\cos(\theta - \phi)\,d\theta - ab\cos(\theta - \phi)\,d(\theta - \phi)$$
$$= 2b\,ds - ab\cos(\theta - \phi)\,d(\theta - \phi).$$

Consequently

$$xdy - ydx = a^2d\theta + b^2d\phi + 2b\,ds - ab\cos(\theta - \phi)\,d(\theta - \phi).$$

But, by Art. 136, the area traced out by B in a complete evolution is represented by $\frac{1}{2}\int(xdy - ydx)$ taken around the ntire curve.

Also, since AC and AB return to their original positions, the integrals of the terms $a^2d\theta$, $b^2d\phi$ and $ab\cos(\theta - \phi)\, d(\theta - \phi)$ disappear; and hence the area in question is equal to bS, where S denotes the entire length registered by the revolving wheel.

Examples.

1. Find the whole area between the curve

$$x^2y^2 + a^2b^2 = a^2y^2$$

and its asymptotes. *Ans.* $2\pi ab$.

2. Find the whole area of the curve

$$a^2y^4 = x^4(a^2 - x^2). \qquad \text{,,} \quad \frac{8a^2}{5}.$$

3. Find the whole area of the curve

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1. \qquad \text{,,} \quad \frac{3}{4}\pi ab.$$

4. Find the whole area included between the folium of Descartes

$$x^3 + y^3 - 3axy = 0$$

and its asymptote. ,, $\frac{3a^2}{2}$.

5. In the logarithmic curve $y = a^x$, prove that the area between the axis of x and any two ordinates is proportional to the difference between the ordinates.

6. Find the area of a loop of the curve

$$r = a\cos n\theta. \qquad \text{,,} \quad \frac{\pi a^2}{n}.$$

7. Find the area of a loop of the curve

$$r = a\cos n\theta + b\sin n\theta. \qquad \text{,,} \quad (a^2 + b^2)\frac{\pi}{n}.$$

The equation of the curve may be written in the form

$$r = \sqrt{a^2 + b^2}\cos(n\theta + a),$$

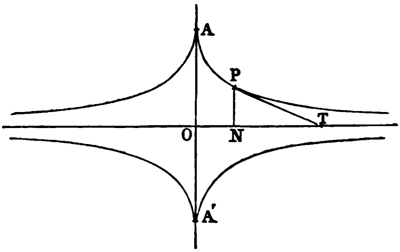
where $\tan a = -\frac{b}{a}$; and consequently its area can be found from the preceding example.

8. Find the area of a loop of the curve

$$r^2 = a^2\cos n\theta + b^2\sin n\theta. \qquad \text{,,} \quad \frac{\sqrt{a^4 + b^4}}{n}.$$

9. Find the area of the tractrix.

The characteristic property of the tractrix is that the intercept on a tangent to the curve between its point of contact and a fixed right line is constant.



Denoting the constant by a, and taking the origin O at the point for which the tangent OA s perpendicular to the axis, we have, P being any point on the curve,

$$PT = a,\ PN = y,$$

$$\frac{dy}{dx} = -\tan PTN = -\frac{y}{\sqrt{a^2-y^2}}$$

$$\therefore\ y\,dx = -\sqrt{a^2-y^2}\,dy.$$

Hence the element of the area of the tractrix is equal to that of a circle of radius a.

It follows immediately that the whole area between the four infinite branches of the tractrix is equal to πa^2. This example furnishes an instance of our being able to determine the area of a curve, from a geometrical property of the curve, without a previous determination of its equation.

If the equation of the tractrix be required, it can be derived from its differential equation

$$dx = -\frac{\sqrt{a^2-y^2}\,dy}{y},$$

from which we get

$$x + \sqrt{a^2-y^2} = a\log\frac{a+\sqrt{a^2-y^2}}{y}.$$

10. If each focal radius vector of an ellipse be produced a constant length c, show that the area between the curve so formed and the ellipse is $\pi c\,(2b+c)$, b being the semi-axis minor of the ellipse.

11. Find the area of a loop of the curve $r^n = a^n\cos n\theta$.

Ans. $\dfrac{a^2\sqrt{\pi}}{2}\,\dfrac{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}\right)}$.

12. If a right line carrying three tracing points A, B, C, move in any manner n a plane, returning to its original position after making a complete revolution; nd if (A), (B), (C) represent the entire areas of the closed curves described by he points A, B, C, respectively, prove that

$$AB\times(C) + BC\times(A) + CA\times(B) + \pi\,.\,AB\,.\,BC\,.\,CA = 0,$$

n which the lines AB, BC, &c., are taken with their proper signs; i. e. $AB = -BA$, &c.

13. If a curve be referred to its radius vector r and the perpendicular p on he tangent, prove that its area is represented by

$$\frac{1}{2}\int\frac{pr\,dr}{\sqrt{r^2-p^2}}.$$

14. Show that the whole area of the inverse of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is represented by

$$\frac{\pi k^4}{\left(1-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}\right)^2}\left\{\frac{1}{a^2}+\frac{1}{b^2}+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}\right)\right\},$$

where α, β are the co-ordinates of the origin of inversion, and k is the radius of the circle of inversion.

15. A given arc of a plane curve turns through a given angle round a fixed point in its plane; what is the area described?

16. A chord of constant length (c) moves about within a parabola, and tangents are drawn at its extremities; find the total area between the parabola and the locus of intersection of the tangents. *Ans.* $\frac{\pi c^3}{2}$.

17. From the centre of an ellipse a tangent is drawn to a semi-circle described on an ordinate to the axis-major; prove that the polar equation of the locus of the point of contact is

$$r^2 = \frac{a^2b^2}{b^2+4a^2\tan^2\theta};$$

and that the whole area of the locus is $\frac{\pi}{2}\,\frac{a^2b}{2a+b}$.

18. Apply the three methods of approximation of Art. 145 to the calculation to 6 decimal places of the definite integral $\int_0^1 \frac{dx}{1+x}$, adopting $\frac{1}{12}$ as the common interval in each case. *Ans.* (1), .693669. (2), .693266. (3), .693224.
The real value of the integral being log 2, or ·693147, to the same number of decimal places.

19. Prove that the sectorial area bounded by two focal vectors r and r' of a parabola is represented by

$$\frac{a^{\frac{1}{2}}}{3}\left\{\left(\frac{r+r'+c}{2}\right)^{\frac{3}{2}}-\left(\frac{r+r'-c}{2}\right)^{\frac{3}{2}}\right\},$$

where c is the chord of the arc, and a the semiparameter.

CHAPTER VIII.

LENGTHS OF CURVES.

147. **Length of Curves referred to Rectangular Axes.**—he usual mode of considering the length of a curve is by eating it as the limit of a polygon when each of its sides is nfinitely small. If the curve be referred to rectangular axes f co-ordinates, the length of the chord joining the points r, y) and $(x+dx, y+dy)$ is $\sqrt{dx^2+dy^2}$, and, consequently, if represent the length of the curve measured from a fixed oint on it, we shall have $ds = \sqrt{dx^2+dy^2}$,

$$s = \int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx,$$

aken between suitable limits.

The value of $\frac{dy}{dx}$ in terms of x is to be got from the equaion of the curve, and thus the finding of s is reducible to a uestion of integration.

The determination of the length of an arc of a curve is alled its *rectification*.

It is evident that if y be taken for the independent variable ve shall have

$$s = \int\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

Again, when x and y are given functions of a single vaiable ϕ, we have

$$s = \int\left\{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2\right\}^{\frac{1}{2}}d\phi.$$

In each case the form of the equation of the curve deternines which of these formulæ should be employed.

The curves whose lengths can be obtained in finite terms are very limited in number. We proceed to consider some of the simplest applications.

148. **The Parabola.**—Writing the equation of the parabola in the form $y^2 = 2mx$, we get $\frac{dx}{dy} = \frac{y}{m}$.

Hence $$s = \frac{1}{m}\int\sqrt{y^2 + m^2}\,dy.$$

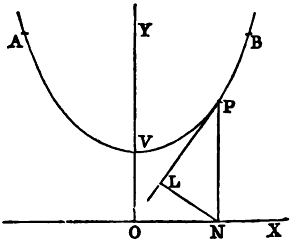
The value of this integral can be obtained from that of the area of an hyperbola (Art. 128), by substituting y for x, and m^2 for $-a^2$.

Thus we have

$$s = \frac{y\sqrt{y^2 + m^2}}{2m} + \frac{m}{2}\log\left(\frac{y + \sqrt{y^2 + m^2}}{m}\right), \qquad (1)$$

the arc being measured from the vertex of the curve.

149. **The Catenary.**—The equation of the catenary, as before observed, is



$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

Hence

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right),$$

$$\frac{ds}{dx} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = \tfrac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

$$\therefore s = \tfrac{1}{2}\int e^{\frac{x}{a}}\,dx + \tfrac{1}{2}\int e^{-\frac{x}{a}}\,dx = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) + \text{const.}$$

If s be measured from the vertex V, we have

$$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right). \qquad (2)$$

Comparing this with Art. 129, we see that

$$\text{area } OVPN = a \times \text{arc } VP.$$

Again, if NL be drawn perpendicular to the tangent PL, we have $\sin LPN = \frac{dx}{ds} = \frac{a}{y}$. Hence $NL = y \sin LPN = a$; and $PL = (y^2 - a^2)^{\frac{1}{2}} = \text{arc } VP$. This result shows that the catenary is the evolute of the tractrix (see Ex. 9, page 199).

150. **Semi-cubical Parabola.**—The equation of this curve is of the form $ay^2 = x^3$.

hence
$$y = \frac{x^{\frac{3}{2}}}{a^{\frac{1}{2}}} \therefore \frac{dy}{dx} = \frac{3}{2}\left(\frac{x}{a}\right)^{\frac{1}{2}}, \quad \frac{ds}{dx} = \left(1 + \frac{9x}{4a}\right)^{\frac{1}{2}},$$

$$\therefore s = \int\left(1 + \frac{9x}{4a}\right)^{\frac{1}{2}} dx = \frac{8a}{27}\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} + \text{const.}$$

If the arc be measured from the vertex, we get

$$s = \frac{8a}{27}\left\{\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} - 1\right\}.$$

151. **Rectification of Evolutes.**—It may be noted that the rectification of the semi-cubical parabola is an immediate consequence of its being the evolute of the ordinary parabola (see Dif. Cal., Art. 233). In like manner the length of any curve can be found if it be the evolute of a known curve, from the property that any portion of the arc of the evolute is the difference between the two corresponding radii of curvature of the curve of which it is the evolute.

For example, we get by this means the lengths of the cycloid, the epicycloid and the hypocycloid.

Again, since the equation of the evolute of an ellipse is

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

the length of any arc of this curve can be at once found.

This can also be readily got otherwise; for, writing the equation in the form

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1,$$

and making $x = a\sin^3\phi$, we get $y = \beta\cos^3\phi$, and

$$ds = (dx^2 + dy^2)^{\frac{1}{2}} = 3\sin\phi\cos\phi\,(a^2\sin^2\phi + \beta^2\cos^2\phi)^{\frac{1}{2}}\,d\phi$$

$$= \frac{3(a^2\sin^2\phi + \beta^2\cos^2\phi)^{\frac{1}{2}}}{2(a^2 - \beta^2)}\,d(a^2\sin^2\phi + \beta^2\cos^2\phi).$$

Hence

$$s = \frac{(a^2\sin^2\phi + \beta^2\cos^2\phi)^{\frac{3}{2}}}{a^2 - \beta^2} + \text{const.}$$

If the arc be measured from the point $x = 0$, $y = \beta$, we get the constant

$$= \frac{-\beta^3}{a^2 - \beta^2}, \text{ and } s = \frac{(a^2\sin^2\phi + \beta^2\cos^2\phi)^{\frac{3}{2}} - \beta^3}{a^2 - \beta^2}.$$

If $a = \beta$, the expression for ds becomes $3a\sin\phi\cos\phi\,d\phi$; hence we get $s = \frac{3}{2}\,a\sin^2\phi$, the arc being taken from the same point as above.

EXAMPLES.

1. Find the length of the logarithmic curve $y = ca^x$.

Here $\log y = x\log a + \log c$, $\therefore \frac{dx}{dy} = \frac{b}{y}$, where $b = \log a$.

Hence
$$s = \int\frac{(b^2 + y^2)^{\frac{1}{2}}\,dy}{y} = \int\frac{y\,dy}{(b^2 + y^2)^{\frac{1}{2}}} + \int\frac{b^2dy}{y\,(b^2 + y^2)^{\frac{1}{2}}}$$

$$= (b^2 + y^2)^{\frac{1}{2}} + b\log\frac{(b^2 + y^2)^{\frac{1}{2}} - b}{y}.$$

2. Find the length of the tractrix.

Here, by definition (see fig. page 199), we have $PT = a$,

$$\therefore \sin PTN = \frac{y}{a}, \quad \text{hence } \frac{ds}{dy} = -\frac{a}{y},$$

$$\therefore s = -a\int\frac{dy}{y} = -a\log y + \text{const.}$$

If the arc be measured from the vertex A, we get

$$\text{arc } AP = a\log\left(\frac{a}{y}\right).$$

3. Find in what cases the curves represented by $a^m y^n = x^{m+n}$ are rectifiable.

Here we have

$$s = \int \left\{ 1 + \left(\frac{m+n}{n} \right)^2 \left(\frac{x}{a} \right)^{\frac{2m}{n}} \right\}^{\frac{1}{2}} dx.$$

Substituting b for $\dfrac{(m+n)^2}{n^2 a^{\frac{2m}{n}}}$, and making $1 + bx^{\frac{2m}{n}} = z^2$, this becomes

$$s = \frac{n}{mb} \int \left(\frac{z^2 - 1}{b} \right)^{\frac{n}{2m} - 1} z^2 dz.$$

This expression is immediately integrable when $\dfrac{n}{2m}$ is a positive integer. Hence if $\dfrac{n}{2m} = r$, the equation becomes of the form $ay^{2r} = x^{2r+1}$.

Again, if $\dfrac{n}{2m}$ be a negative integer, the expression under the integral sign becomes rational, and can accordingly be integrated. This leads to the form $y^{2r} = ax^{2r-1}$. Accordingly, all curves comprised in the equation $ay^m = x^{m\pm 1}$ are rectifiable, m being any integer. (Compare Art. 58).

152. **The Ellipse.**—The simplest expression for the arc of an ellipse is obtained by taking $x = a \sin \phi$, whence

$$y = b \cos \phi, \text{ and } ds = (a^2 \cos^2\phi + b^2 \sin^2\phi)^{\frac{1}{2}} d\phi,$$

$$\therefore s = \int (a^2 \cos^2\phi + b^2 \sin^2\phi)^{\frac{1}{2}} d\phi.$$

It is often more convenient to write this in the form

$$s = a \int (1 - e^2 \sin^2\phi)^{\frac{1}{2}} d\phi,$$

e being the eccentricity of the ellipse.

It may be observed that ϕ is the complement of the *eccentric* angle belonging to the point (x, y).

The length of an elliptic quadrant is represented by the definite integral $a \int_0^{\frac{\pi}{2}} (1 - e^2 \sin^2\phi)^{\frac{1}{2}} d\phi$.

We postpone the further consideration of elliptic arcs to a subsequent part of the chapter.

153. **Rectification in Polar Co-ordinates.**—If the curve be referred to polar co-ordinates we plainly have (Dif. Cal., Art. 176) $ds^2 = dr^2 + r^2 d\theta^2$; hence we get

$$s = \int\Bigg(r^2 + \left(\frac{dr^2}{d\theta^2}\right)^{\frac{1}{2}} d\theta, \quad \text{or } s = \int\left(1 + \frac{r^2 d\theta^2}{dr^2}\right)^{\frac{1}{2}} dr.$$

For example, the length of the spiral of Archimedes, $r = a\theta$, is given by the equation

$$s = \frac{1}{a}\int(r^2 + a^2)^{\frac{1}{2}}\, d\theta.$$

Comparing this with the formula (1) for the parabola, it follows that the length of any arc of the spiral, measured from its pole, is equal to that of a parabola measured from its vertex.

EXAMPLES.

1. Cardioid, $\qquad r = a(1 + \cos\theta).$

Here $\frac{dr}{d\theta} = -a\sin\theta$, and hence

$$s = a\int\{(1 + \cos\theta)^2 + \sin^2\theta\}^{\frac{1}{2}}\, d\theta = 2a\int\cos\frac{\theta}{2}\, d\theta = 4a\sin\frac{\theta}{2} + \text{constant.}$$

The constant becomes zero if we measure s from the point for which $\theta = 0$.

2. Logarithmic spiral, $\qquad r = a^\theta.$

Here, if $b = \log a$, we get

$$\frac{rd\theta}{dr} = \frac{1}{\log a}, \quad \therefore s = \int_{r_0}^{r_1}\left(1 + \frac{1}{b^2}\right)^{\frac{1}{2}} dr = \frac{(1 + b^2)^{\frac{1}{2}}}{b}(r_1 - r_0).$$

Accordingly, the length of any arc is proportional to the difference between the vectors of its extremities; a result which also follows immediately from the property that the curve cuts its radius vector at a constant angle.

3. $\qquad r^m = a^m \cos m\theta.$

Taking the logarithmic differentials, we get $\frac{dr}{rd\theta} = -\tan m\theta$;

$$\therefore \frac{ds}{rd\theta} = \sec m\theta.$$

Hence $\qquad s = a\int\cos^{\frac{1}{m}-1} m\theta\, d\theta.$

Or, writing ϕ for $m\theta$,

$$s = \frac{a}{m}\int\cos^{\frac{1}{m}-1}\phi\, d\phi.$$

readily integrated when $\frac{1}{m}$ is an integer (see Art. 53).

atever be the value of m we can express the complete length of a loop of ve in Gamma-functions. For if we integrate between 0 and $\frac{\pi}{2}$, we ob- get the length of half the loop.
ice the length of the loop (Art. 120) is

$$\frac{a\sqrt{\pi}}{m}\,\frac{\Gamma\left(\frac{1}{2m}\right)}{\Gamma\left(\frac{m+1}{2m}\right)}.$$

4. **Formula of Legendre on Rectification.**—Another for- of considerable utility in rectification follows imme- y from the result obtained in Art. 188, Dif. Cal. For, result be written in the form

$$\frac{d(s-t)}{d\omega} = p, \text{ we get } s - t = \int p\,d\omega.$$

quently, the total increment of $s - t$ between any two on a curve is equal to $\int p\,d\omega$ taken between the same oints.

›r example, in the parabola we have $p = \frac{a}{\cos\omega}$, and

$$s - t = a\int\frac{d\omega}{\cos\omega} = a\log\tan\left(\frac{\pi}{4} + \frac{\omega}{2}\right) + \text{const.}$$

we measure the arc from the vertex of the curve, and ʼe that $t = \frac{dp}{d\omega}$, this gives

$$s = \frac{a\sin\omega}{\cos^2\omega} + a\log\tan\left(\frac{\pi}{4} + \frac{\omega}{2}\right).$$

ıe student can without difficulty identify this result with iven in Art. 148.

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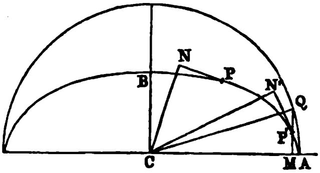
his theorem is due to Legendre. See Traité des Fonctions Elliptiques. ., p. 588.

155. **Application to Ellipse. Fagnani's Theorem.**—In the ellipse we have

$$p^2 = a^2 \cos^2\omega + b^2 \sin^2\omega.$$



Hence, measuring the arc from the vertex A, and observing that in this case PN is to be taken with a negative sign, we have

$$\text{arc } AP + PN = \int_0^a (a^2 \cos^2\omega + b^2 \sin^2\omega)^{\frac{1}{2}}\, d\omega,$$

where $a = \angle ACN$.

But, in Art. 152, we have found that if ϕ be measured from the vertex B, the arc is represented by

$$\int (a^2 \cos^2\phi + b^2 \sin^2\phi)^{\frac{1}{2}}\, d\phi.$$

Consequently if we make $\angle BCQ = a = \angle ACN$, and draw QM perpendicular to the axis major meeting the curve in P', we shall have

$$\text{arc } BP' = \text{arc } AP + PN,$$

or, taking away the common arc PP',

$$BP - AP' = PN. \qquad (3)$$

This remarkable result is known as Fagnani's theorem,[*] and shows that we can in an indefinite number of ways find two arcs of an ellipse whose difference is expressible by a right line.

We add a few properties connecting the points P and P' in this construction.

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[*] Fagnani. Produzioni Mathematice. 1750. It may be noted that if we integrate the equation of Art. 114, Dif. Cal., taking the angle C as obtuse, and adopting zero for the lowest limit in each integral, we obtain

$$\int_0^a \sqrt{1 - k^2 \sin^2 a}\, da + \int_0^b \sqrt{1 - k^2 \sin^2 b}\, db - \int_0^c \sqrt{1 - k^2 \sin^2 c}\, dc$$
$$= k^2 \sin a \sin b \sin c,$$

where a, b, c are connected by the relation

$$\cos c = \cos a \cos b - \sin a \sin b \sqrt{1 - k^2 \sin^2 c}.$$

The negative sign is given to $\sqrt{1 - k^2 \sin^2 c}$ since $\cos C$ is negative in this case.

This equation furnishes a relation between three elliptic arcs, from which Fagnani's theorem can be readily deduced, as well as many other theorems conne[illegible]h arcs. See Legendre. Fonc. Ellip. Tome i., ch. 9.

EXAMPLES.

1. If (x, y) and (x', y') be the co-ordinates of P and P', respectively; prove the following:—

$$(1).\ PN = \frac{e^2 xx'}{a}, \qquad (2)\ PN = P'N', \qquad (3)\ CN \,.\, CN' = CA \,.\, CB,$$
$$(4)\ CP^2 + CN'^2 = CA^2 + CB^2.$$

2. Divide an elliptic quadrant into two parts whose difference shall be equal to the difference of the semiaxes.

This has place when P and P' coincide; in which case $CN = \sqrt{ab}$, and $PN = a - b$.

We shall designate the point so determined on the elliptic quadrant as Fagnani's point.

3. Show that if a tangent be drawn at Fagnani's point, the intercepts between its point of contact and its points of intersection with the axes are respectively equal in length to the semi-axes of the ellipse.

4. If the lines PN and $P'N'$ be produced to meet, show that they intersect on the confocal hyperbola which passes through the points of intersection of the tangents to the ellipse at its vertices. Show also that this hyperbola cuts the ellipse in Fagnani's point.

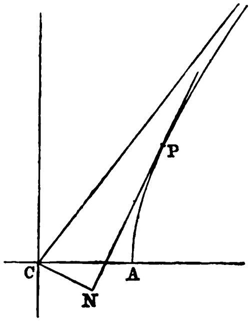
156. **The Hyperbola.**—In the hyperbola we have

$$p^2 = a^2 \cos^2\omega - b^2 \sin^2\omega.$$

Hence, measuring the arc from the vertex A of the curve, we find, since ω is measured below the axis,

$$PN - AP = \int_0^{\alpha} (a^2 \cos^2\omega - b^2 \sin^2\omega)^{\frac{1}{2}}\, d\omega, \qquad (4)$$

where $\alpha = \angle ACN$.



As we proceed along the hyperbola the perpendicular p diminishes, and vanishes when the tangent becomes the asymptote.

Moreover, as the limit of ω in this case becomes $\tan^{-1}\dfrac{a}{b}$, it follows that the difference between the asymptote and the infinite hyperbolic arc, measured from the vertex, is represented by the definite integral

$$\int_0^{\tan^{-1}\frac{a}{b}} (a^2 \cos^2\omega - b^2 \sin^2\omega)^{\frac{1}{2}}\, d\omega.$$

EXAMPLES.

1. If $a > b$, prove that

$$\int (a + b \cos \phi)^{\frac{1}{2}} d\phi$$

is represented by an elliptic arc, and that the semiaxes of the ellipse are the greatest and least values of $(a + b \cos \phi)^{\frac{1}{2}}$.

2. If $a \angle b$, prove that

$$\int (a + b \cos \phi)^{\frac{1}{2}} d\phi$$

is represented by the difference between a right line and a hyperbolic arc.

157. **Landen's Theorem on a Hyperbolic Arc.**—We next proceed to establish an important theorem, due to Landen;* namely, that *any arc of a hyperbola can be expressed in terms of the arcs of two ellipses.*

This can be easily seen as follows:—In any triangle, adopting the usual notation, we have

$$c = a \cos B + b \cos A.$$

Now, representing by C the external angle at the vertex C, we have $C = A + B$, and hence

$$cdC = (a \cos B + b \cos A)\, dA + (a \cos B + b \cos A)\, dB.$$

Consequently, supposing the sides a and b constant, and the remaining parts variable, we have

$$\int cdC = \int a \cos B dA + \int b \cos A dB + 2a \sin B + \text{const.},$$

or

$$\int \sqrt{a^2 + b^2 + 2ab \cos C}\, dC = \int \sqrt{a^2 - b^2 \sin^2 A}\, dA + \int \sqrt{b^2 - a^2 \sin^2 B}\, dB$$
$$+ 2a \sin B + \text{const.} \qquad (5)$$

Now, if we suppose $a > b$, $\int \sqrt{a^2 - b^2 \sin^2 A}\, dA$ represents (Art. 152) the arc of an ellipse, of axis major $2a$, and eccentricity $\frac{b}{a}$ Also $\int \sqrt{b^2 - a^2 \sin^2 B}\, dB$ represents (Art. 156) the difference between a right line and the arc of a hyperbola, whose axis major is b and eccentricity $\frac{a}{b}$.

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* Landen. Philosophical Transactions, 1775.

Again $\sqrt{a^2+b^2+2ab\cos C} = \sqrt{(a-b)^2\sin^2\frac{C}{2}+(a+b)^2\cos^2\frac{C}{2}}$,

and consequently the integral

$$\int\sqrt{a^2+b^2+2ab\cos C}\,dC$$

represents an arc of the ellipse whose semiaxes are $a+b$ and $a-b$.

Hence, Landen's theorem follows immediately.

It should be noted that the limiting values of A, B and C are connected by the relations

$$a\sin B = b\sin A, \text{ and } C = A + B.$$

Again, if we suppose the angle A to increase from 0 to π, the external angle C will increase at the same time from 0 to π, while B will commence by increasing from 0 to α, and afterwards diminish from α to 0, $\left(\text{where } \alpha = \sin^{-1}\frac{b}{a}\right)$. Moreover, in the latter stage $b\cos A$ is negative, and dB also negative, consequently the term $b\cos A\,dB$ is positive throughout the entire integration; and the total value of

$$\int\sqrt{b^2-a^2\sin^2 B}\,dB \text{ is represented by } 2\int_0^\alpha\sqrt{b^2-a^2\sin^2 B}\,dB\cdot$$

Hence, substituting ϕ for $\frac{C}{2}$, and integrating between the limits indicated, we get, after dividing by 2,

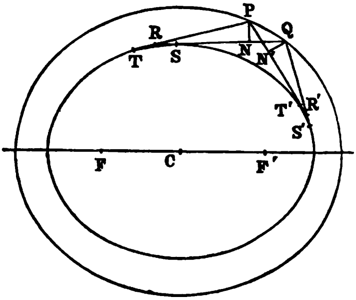
$$\int_0^{\frac{\pi}{2}}\{(a+b)^2\sin^2\phi + (a-b)^2\cos^2\phi\}^{\frac{1}{2}}\,d\phi$$

$$= \int_0^{\frac{\pi}{2}}(a^2 - b^2\sin^2 A)^{\frac{1}{2}}\,dA + \int_0^\alpha(b^2 - a^2\sin^2 B)^{\frac{1}{2}}\,dB. \qquad (6)$$

Accordingly, *the difference between the length of the asymptote and of the infinite arc of a hyperbola is equal to the difference between two elliptic quadrants.* This result is also due to Landen.

We next proceed to two important theorems which may be regarded as extensions of Fagnani's theorem.

158. **Theorem* of Dr. Graves.**—If from any point P on the exterior of two confocal ellipses, tangents PT and PT' be drawn to the interior, then the difference $PT + PT' - TT'$ between the sum of the tangents and the arc between their points of contact is constant.



For, draw the tangents QS and QS' from a point Q, regarded as infinitely near to P, and drop the perpendiculars PN and QN'; then, since the conics are confocal, we have

$$\angle PQN = \angle QPN', \quad \therefore PN' = QN.$$

Also $$PT = TR + RN = TR + RS + SN = TS + SN$$
$$= TS + SQ - QN.$$

In like manner

$$PT' = PN' + S'Q - T'S',$$
$$\therefore PT + PT' = QS + QS' + TS - T'S',$$

or $$PT + PT' - TT' = QS + QS' - SS'.$$

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* This elegant theorem was arrived at by Dr. Graves for the more general case of spherical conics from the reciprocal theorem, viz. If two spherical conics have the same cyclic arcs, then any arc touching the inner will cut from the outer a segment of constant area. (See Graves' translation of Chasles on Cones and Spherical Conics, p. 77, Dublin, 1841).

It should be remarked that the theorems of this and of the following article were investigated independently by M. Chasles. The student will find in the *Comptes Rendus*, 1843, 1844, a number of beautiful applications by that great geometrician of these theorems, as well to properties of confocal conics, as also to the addition of elliptic functions of the first species.

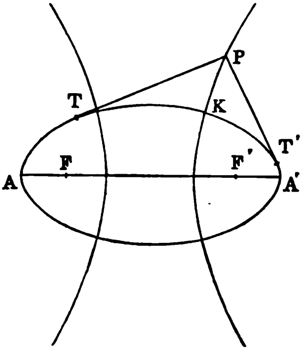
Hence, $PT + PT' - TT'$ does not change in passing to the consecutive point Q; which proves the proposition.

The analogous theorem, due to Professor Mac Cullagh, may be stated as follows:—

159. **Theorem.**—If tangents PT, PT' be drawn to an ellipse from any point on a confocal hyperbola, then the difference of the tangents is equal to the difference of the arcs TK and KT'.

The proof is left to the student, and is nearly identical with that given for the previous theorem.

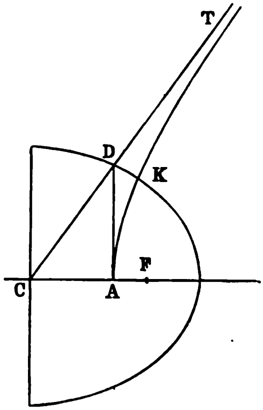
This result still holds when the tangents are drawn from a point on an ellipse to a confocal hyperbola, provided that the tangents both touch the same branch of the hyperbola; as can be seen without difficulty.



As an application* we shall prove another theorem of Landen; viz., that the *difference between the length of the asymptote and of the infinite branch of a hyperbola can be expressed in terms of an arc of the hyperbola.*

For, let the tangent at A meet the asymptote in D, and suppose a confocal ellipse drawn through D. Then, regarding DT as a tangent to the hyperbola, it follows, by the theorem just established, that the difference between DT and KT is equal to the difference between DA and AK.

Consequently the difference between the asymptote CT and the hyperbolic branch AT is equal to $DA + DC - 2KA$. Consequently the required difference is expressible in terms of given lines and of the hyperbolic arc AK.



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[*] I am indebted to Dr. Ingram for this application of Prof. M'Cullagh's theorem.

We next proceed to consider two important curves whose rectification depends on that of the ellipse.

160. **The Limaçon.**—From the equation of the limaçon

$$r = a\cos\theta + b, \text{ we get } \frac{dr}{d\theta} = -a\sin\theta,$$

and hence

$$ds = (a^2 + b^2 + 2ab\cos\theta)^{\frac{1}{2}}\,d\theta.$$

$$\therefore\ s = \int\left\{(a+b)^2\cos^2\frac{\theta}{2} + (a-b)^2\sin^2\frac{\theta}{2}\right\}^{\frac{1}{2}} d\theta.$$

Accordingly, the rectification of the limaçon depends on that of the ellipse whose semiaxes are $a + b$ and $a - b$.

161. **The Epitrochoid and Hypotrochoid.**—The epitrochoid is represented by the equations (See Dif. Cal. Art. 272)

$$x = (a+b)\cos\theta - c\cos\frac{a+b}{b}\theta$$

$$y = (a+b)\sin\theta - c\sin\frac{a+b}{b}\theta.$$

Hence

$$\frac{dx}{d\theta} = -(a+b)\left\{\sin\theta - \frac{c}{b}\sin\frac{a+b}{b}\theta\right\}$$

$$\frac{dy}{d\theta} = (a+b)\left\{\cos\theta - \frac{c}{b}\cos\frac{a+b}{b}\theta\right\}.$$

Squaring and adding we get

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{a+b}{b}\right)^2\left\{b^2 + c^2 - 2bc\cos\frac{a\theta}{b}\right\}$$

$$\therefore\ s = \frac{a+b}{b}\int\left(b^2 + c^2 - 2bc\cos\frac{a\theta}{b}\right)^{\frac{1}{2}} d\theta.$$

Hence, substituting $\frac{2b\phi}{a}$ for θ, we get

$$s = \frac{2(a+b)}{a}\int\{(b+c)^2\sin^2\phi + (b-c)^2\cos^2\phi\}^{\frac{1}{2}}\,d\phi.$$

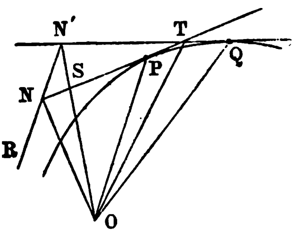
Consequently the length of an arc of the epitrochoid is equal to that of an ellipse.

The corresponding form for the hypotrochoid is obtained by changing the sign of b.

162. **Steiner's Theorem on Rectification of Roulettes.**—If any curve roll on a right line, the length of the arc of the roulette described by any point is equal to that of the corresponding arc of the pedal, taken with respect to the generating point as origin.

For (see figure, Art. 142), the element OO' of the roulette is equal to $OPd\omega$.

Again, to find the element of the pedal. Since the angles at N and N' are right, the quadrilateral $NN'TO$ is inscribable in a circle, and consequently $NN' = OT \sin NON'$. But, in the limit, NN' becomes the element of the pedal, and OT becomes OP: hence the element of pedal is $OPd\omega$; consequently the element of the pedal is equal to the corresponding element of the roulette, ∴ &c.



We proceed to point out a few elementary examples of this principle. In the first place, it follows that the length of an arc of the cycloid is the same as that of the cardioid; and the length of the trochoid as that of the limaçon. Again, if an ellipse roll on a right line the length of the roulette described by either focus is equal to the corresponding arc of the auxiliary circle.

Moreover, it is easily seen, as in Art. 143, that, if one curve roll on another, the elements ds and ds', of the roulette, and of the corresponding pedal are connected by the relation

$$ds = ds'\left(1 + \frac{\rho}{\rho'}\right).$$

In the case of one circle rolling on another, this relation shows that the arcs of epicycloids and of epitrochoids are proportional to the arcs of cardioids and of limaçons: which agrees with the results established already. For further development of this method the student is referred to Mr. Besant's "Notes on Roulettes and Glissettes."

We next proceed to the rectification of the Oval of Descartes.

163. **Oval of Descartes.**—This curve is defined as the locus of a point whose distances, r and r', from two fixed points are connected by the equation

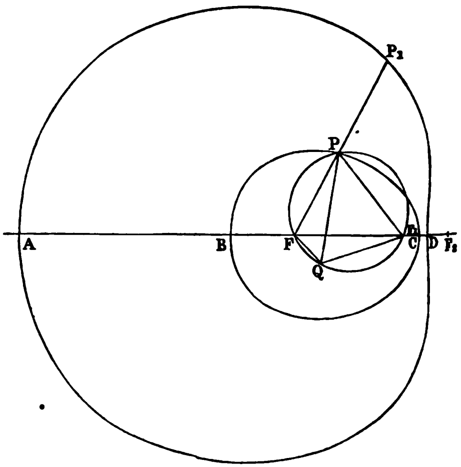
$$lr + mr' = d,$$

where l, m, d are given constants.

For convenience we shall write the equation in the form

$$lr + mr' = nc, \tag{7}$$

where c is the distance between the fixed points.



The polar equation of the curve is easily got. For, let F and F_1 be the fixed points, and $\angle F_1FP = \theta$, then we have r'^2

$= r^2 + c^2 - 2rc\cos\theta$;

also from (7),

$$m^2r'^2 = (nc - lr)^2,$$

hence the polar equation of the locus is readily seen to be

$$r^2 - 2rc\frac{ln - m^2\cos\theta}{l^2 - m^2} + c^2\frac{n^2 - m^2}{l^2 - m^2} = 0. \tag{8}$$

For simplicity, we shall write this in the form

$$r^2 - 2r\Omega + C = 0. \tag{9}$$

Solving this equation for r, we get

$$r = \Omega \pm \sqrt{\Omega^2 - C}, \text{ or } FP_1 = \Omega + \sqrt{\Omega^2 - C},\ FP = \Omega - \sqrt{\Omega^2 - C}.$$

It can be seen without difficulty that, so long as l, m, n are real and unequal,* the curve consists of two ovals, one lying inside the other, as in the figure.

Again we get from (9), by differentiation,

$$(r-\Omega)\,dr = r\Omega' d\theta, \quad \text{where } \Omega' = \frac{d\Omega}{d\theta};$$

$$\therefore \frac{dr}{rd\theta} = \frac{\Omega'}{r-\Omega} = \frac{\Omega'}{\sqrt{\Omega^2 - C}}; \text{ hence } \frac{ds}{rd\theta} = \frac{\sqrt{\Omega^2+\Omega'^2-C}}{\sqrt{\Omega^2 - C}}.$$

$$\text{Or} \quad ds = \frac{\Omega\sqrt{\Omega^2+\Omega'^2-C}\,d\theta}{\sqrt{\Omega^2-C}} \pm \sqrt{\Omega^2+\Omega'^2-C}\,d\theta, \quad (10)$$

the upper sign corresponding to the outer oval, and the lower to the inner.

Hence the difference between the two corresponding elementary arcs is equal to

$$2\sqrt{\Omega^2+\Omega'^2-C}\,d\theta, \quad \text{or } 2\sqrt{a^2+2ab\cos\theta+b^2-C}\,d\theta,$$

(writing Ω in the form $a + b\cos\theta$), which plainly represents the element of an ellipse. Consequently, the difference between two corresponding arcs of the ovals can be represented by the arc of an ellipse. This remarkable theorem is due to Mr. W. Roberts (Liouville, xv., p. 195). Some years after its publication it was shown by Professor Genocchi (Tortolini, 1864, p. 97) that the arc* of a Cartesian is expressible in terms of three elliptic arcs.

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* It may be remarked that equation (7) contains the circle, the ellipse or hyperbola, and the limaçon, as particular cases.

Thus, (1) if $n = 0$, the equation becomes $lr + mr' = 0$, which represents a circle.

(2) If $l = \pm m$, the locus is an ellipse when $n > l$, and an hyperbola when $n < l$.

(3) If either l or m be equal to n, the locus is a limaçon.

The points F and F_1 are called foci of the oval, and possess many properties analogous to those of the foci of a conic. They are connected with the points A, B, C, D by very simple relations. For, from (8) we have $FP \,.\, FP_1 = c^2 \dfrac{n^2 - m^2}{l^2 - m^2}$ = a constant. Hence $FA \,.\, FB = FC \,.\, FD$. In like manner, we can see that $F_1 B \,.\, F_1 D = F_1 A \,.\, F_1 C$. Consequently, the foci F, F_1 can be found by a simple geometrical construction when the four points A, B, C, D are given. Again, if F_2 be found, so as to satisfy the analogous relation $F_2 A \,.\, F_2 D = F_2 B \,.\, F_2 C$, it is a *third* focus of the curve, and it can be shown to possess the same properties as F and F_1; for example, the distances from any point P to F and F_2 are connected by an equation similar to (7).

It should be noted that either oval is the inverse of the other with respect to either of the foci F_1 F_1; while with respect to the focus F_2 each oval is its own inverse.

In order to establish this result, we commence by proving one or two elementary properties of the curve.

Suppose a circle described through F, F_1, and P; and let PQ be the normal at P to the oval, meeting the circle in Q, and join FQ and F_1Q; then let $\angle FPQ = \omega$, and $F_1PQ = \omega'$; and since $l\dfrac{dr}{ds} + m\dfrac{dr'}{ds} = 0$, we have $l \sin \omega = m \sin \omega'$,

$$\therefore \quad F_1Q : FQ = l : m.$$

Also, since $lr + mr' = nc$; and (by Ptolemy's theorem)

$$FP \,.\, F_1Q + F_1P \,.\, FQ = FF_1 \,.\, PQ,$$

we have

$$\frac{F_1Q}{l} = \frac{FQ}{m} = \frac{PQ}{n}.$$

Hence, denoting the common value of these fractions by u, we have

$$F_1Q = lu, \quad FQ = mu, \quad PQ = nu.$$

Again

$$\tan \omega = \frac{dr}{r d\theta} = \frac{\Omega'}{\sqrt{\Omega^2 - C}}, \quad \therefore \cos \omega = \frac{\sqrt{\Omega^2 - C}}{\sqrt{\Omega^2 + \Omega'^2 - C}}.$$

Hence the first term in the expression for ds in (10) is equal to

$$\frac{\Omega d\theta}{\cos \omega} = \frac{c}{l^2 - m^2}\,\frac{ln - m^2 \cos \theta}{\cos \omega}\,d\theta.$$

Again, let $\quad \angle FPF_1 = \psi, \quad \angle PF_1C = \phi,$

and we have the two following relations between the angles θ, ϕ, ψ,

$$\phi = \theta + \psi, \quad l \sin \phi + m \sin \theta = n \sin \psi. \qquad (11)$$

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* For the proof of this theorem given in the text I am indebted to Mr. Panton.

Hence

$$d\phi - d\theta = d\psi,\ l\cos\phi d\phi + m\cos\theta d\theta = n\cos\psi d\psi,$$

$$\therefore (ln - m^2\cos\theta)\, d\theta = l\,(n + m\cos\phi)\, d\phi - n\,(l + m\cos\psi)\, d\psi,$$

or

$$\frac{ln - m^2\cos\theta}{\cos\omega}\, d\theta = l\,\frac{n + m\cos\phi}{\cos\omega}\, d\phi - n\,\frac{l + m\cos\psi}{\cos\omega}\, d\psi. \quad (12)$$

Again, from the triangle *FPQ*, we have

$$r\cos\omega = PQ + FQ\cos\phi = (n + m\cos\phi)\, u,$$

$$\therefore \frac{n + m\cos\phi}{\cos\omega} = \frac{r}{u} = \sqrt{m^2 + n^2 + 2mn\cos\phi}.$$

In like manner it can be seen that

$$\frac{l + m\cos\psi}{\cos\omega} = \sqrt{l^2 + m^2 + 2\,lm\cos\psi}.$$

Hence we have

$$\int\frac{\Omega\, d\theta}{\cos\omega} = \frac{lc}{l^2 - m^2}\int\sqrt{m^2 + n^2 + 2mn\cos\phi}\, d\phi$$

$$- \frac{nc}{l^2 - m^2}\int\sqrt{l^2 + m^2 + 2lm\cos\psi}\, d\psi, \quad (13)$$

each of which latter integrals represents the arc of an ellipse: and, consequently, the arc of a Cartesian oval is expressible in the required manner.

It should be noted that the limiting values of θ, ϕ, and ψ are connected by the relations given in (11).

To complete the proof it remains to show that the axes of the ellipses are the lines (*AB*, *CD*), (*AC*, *BD*), and (*AD*, *BC*), respectively: a result also given by Sig. Genocchi. First, with respect to the ellipse whose element is $\sqrt{\Omega^2 + \Omega'^2 - C}\, d\theta$, it is plain that its axes are the greatest and least values of $2\sqrt{\Omega^2 + \Omega'^2 - C}$, or of $2\sqrt{a^2 + b^2 + 2ab\cos\theta - C}$: but these are $2\sqrt{(a + b)^2 - C}$ and $2\sqrt{(a - b)^2 - C}$, which are plainly the same as the greatest and least values of PP_1; and, consequently, are AB and CD.

Again, from the equation $lr + mr' = nc$, we get

$$lFB + m(FB + c) = nc. \quad \therefore FB = \frac{(n - m)c}{l + m}.$$

In like manner

$$FC = \frac{(n + m)c}{l + m}.$$

Also, since we get the points on the outer oval by changing the sign of m, we have

$$FA = \frac{(n + m)c}{l - m}, \quad FD = \frac{(n - m)c}{l - m}:$$

and, consequently

$$AD = \frac{2nc}{l - m}, \quad BC = \frac{2nc}{l + m},$$

$$AC = \frac{2lc(n + m)}{l^2 - m^2}, \quad BD = \frac{2lc(n - m)}{l^2 - m^2};$$

but these are readily seen to be the values for the axes of the ellipses in (13).

164. **Rectification of Curves of double Curvature.**—If the points in a curve be not situated in the same plane, the curve is said to be one of double curvature. The expression for its length is obtained in an analogous manner to that adopted for plane curves: for, if we refer the curve to a system of rectangular axes in space, and denote the co-ordinates of two consecutive points by (x, y, z), $(x + dx, y + dy, z + dz)$, we get for the element of length, ds, the value

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

The curve is commonly supposed to be determined by the intersection of two cylindrical surfaces, whose equations are denoted by

$$f(x, y) = 0, \qquad \phi(x, z) = 0.$$

From these equations, if $\frac{dy}{dx}$ and $\frac{dz}{dx}$ be determined, the formula of rectification is

$$s = \int \left\{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}} dx. \qquad (14)$$

When z is taken as the independent variable, this formula becomes

$$s = \int \left\{1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2\right\}^{\frac{1}{2}} dz:$$

the limits being in each case determined by the conditions of the question.

The simplest example is that of the helix, or the curve formed by the thread of a screw. From its mode of generation it is easily seen that the helix is represented by two equations of the form

$$x = a\cos\left(\frac{z}{b}\right), \qquad y = a\sin\left(\frac{z}{b}\right).$$

Hence

$$\frac{dx}{dz} = -\frac{a}{b}\sin\left(\frac{z}{b}\right), \qquad \frac{dy}{dz} = \frac{a}{b}\cos\left(\frac{z}{b}\right).$$

$$\therefore\ ds = \left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}} dz, \text{ or } \quad s = \left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}} z;$$

he arc being measured from the point in which the helix neets the plane of xy.

EXAMPLES.

1. Find the length of the curve whose equations are

$$y = \frac{x^2}{2a}, \quad z = \frac{x^3}{6a^2}.$$

Here $ds = \left(1 + \frac{x^2}{a^2} + \frac{x^4}{4a^4}\right)^{\frac{1}{2}} dx = \left(1 + \frac{x^2}{2a^2}\right) dx = x + \frac{x^3}{6a^2} = x + z:$

he arc being measured from the origin.

This is a case of a system of curves which are readily rectified; for, in general, whenever

$$\left(\frac{dy}{dx}\right)^2 = 2\frac{dz}{dx},$$

we have
$$\left(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}\right)^{\frac{1}{2}} = \left(1 + \frac{dz}{dx}\right),$$

and therefore
$$ds = dx + dz, \quad \text{or } s = x + z + \text{const.}$$

Thus, if $y = f(x)$ be one of the equations of a curve, we get $\frac{dy}{dx} = f'(x)$, and hence, if a second equation be determined from the equation

$$\frac{dz}{dx} = \frac{1}{2}\left\{f'(x)\right\}^2,$$

the length of the curve is represented by $x + z +$ const.: the value of the constant being determined by the conditions of the problem.

For instance, if $y = a \sin x$, we get $f'(x) = a \cos x$, and

$$\frac{dz}{dx} = \frac{a^2}{2}\cos^2 x, \qquad \therefore\ z = \frac{a^2}{4}(x + \cos x \sin x).$$

hence the length of the curve of intersection of the cylindrical surfaces

$$y = a \sin x, \quad z = \frac{a^2}{4}\ (x + \cos x \sin x)$$

is $z + x$; the length being measured from the origin.

2. $y = 2\sqrt{ax} - x, \quad z = x - \frac{2}{3}\sqrt{\frac{x^3}{a}}.$ *Ans.* $s = x + y - z.$

3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad x = \frac{a}{2}\left(e^{\frac{z}{a}} + e^{-\frac{z}{a}}\right)$, the length being measured from the point of intersection of the curve with the plane of xy.

Ans. $s = \frac{(a^2 + b^2)^{\frac{1}{2}}}{a}(x^2 - a^2)^{\frac{1}{2}}.$

EXAMPLES.

1. Find the length of any arc of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),$$

and show that the area between the curve, the axis of x, and the ordinates at two points on the curve, is equal to a times the length of the arc terminated by those points.

2. In any curve prove that $s = \int \frac{r\,dr}{\sqrt{r^2 - p^2}}$, and hence find the length of a parabolic arc.

3. Show that the integral $\int \frac{x\,dx}{\sqrt{bx^2 - x^4 - c^2}}$ may be represented by an arc of a circle, and find the limiting values of x for its possibility.

4. Show that the length of an elliptic arc is represented by $\int \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}}\,dx$, where a is the semiaxis major, and e the eccentricity.

5. Express the length of an elliptic quadrant in a series of ascending powers of its eccentricity.

$$\textit{Ans. } \frac{\pi a}{2}\left\{1 - \left(\frac{1}{2}\right)^2 \frac{e^2}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{e^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{e^6}{5} - \&c.\right\}.$$

6. Prove that the integral

$$\frac{x^2\,dx}{\sqrt{(x^2 - \beta^2)(a^2 - x^2)}}$$

can be represented by an arc of the ellipse whose semiaxes are a and β.

7. Show that the rectification of the sinusoid $y = b \sin x$ is the same as that of an ellipse.

8. Prove that the whole length of the *first negative pedal* of an ellipse, taken with respect to a focus, is equal to the circumference of the circle described on the axis minor as diameter.

9. Show that the length of an arc of the curve $r = a \sin n\theta$ is equal to that of an arc of the ellipse whose semiaxes are a and na.

10. If from the equation of a curve referred to rectangular co-ordinates, we form an equation in polar co-ordinates, by taking $r = y$ and $r\,d\theta = dx$, then the lengths of the corresponding arcs of the two curves are equal, and the area $\int y\,dx$ of the former curve is equal to the corresponding sectorial area of the latter.

11. Prove that the difference between the lengths of the two loops of the limaçon $r = a \cos\theta + b$ is equal to $8b$: a being greater than b.

12. Being given three points A, B, C on the circumference of an ellipse, show that we can always find, at either side of C, a fourth point D such that the difference between AB and CD shall be equal to a right line.

13. If a circle be described touching two tangents to an ellipse and also touching the ellipse, prove that the point of contact with the ellipse divides the elliptic arc between the points of contact of the tangents into two parts, whose difference is equal to the difference of the lengths of the tangents (Chasles, Comptes Rendus, 1843).

14. Prove that the entire length of any closed curve is represented by $\int \frac{pds}{\rho}$ taken round the entire curve; ρ being the radius of curvature at any point, and p the length of the perpendicular from any fixed point on the tangent.

15. If $e^y = \frac{e^x + 1}{e^x - 1}$ be the equation of a curve, prove that $\frac{ds}{dx} = \frac{e^{2x} + 1}{e^{2x} - 1}$, and hence rectify the curve.

16. Calculate approximately, by the tables of Art. 123, the whole length of a loop of the curve $r^{\frac{4}{5}} = a^{\frac{4}{5}} \cos \frac{4}{5} \theta$.

Here, by Ex. 3, p. 207, the required length is

$$\frac{5}{4} a\sqrt{\pi} \frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{9}{8}\right)} \text{ or } 2a\sqrt{\pi} \frac{\Gamma\left(\frac{13}{8}\right)}{\Gamma\left(\frac{9}{8}\right)}.$$

Hence, taking logarithms, and observing that $\frac{13}{8} = 1.625$, and $\frac{9}{8} = 1.125$, we get as the required approximation $a \times 3.29488$. The figure of this curve is exhibited in Art. 261, Dif. Cal.

17. In a Cartesian Oval whose two internal foci coincide, prove that the difference of the two arcs, intercepted by any two transversals from the external focus, is equal to a straight line which may be found. [The above curve is the inverse of an ellipse from a focus].—Professor Crofton, *Educ. Times, June*, 1874.

From (8) page 219 it follows, making $n = l$, that the equation of the limaçon, in this case, is

$$r^2 - 2rc \frac{m^2 \cos \theta - l^2}{m^2 - l^2} + c^2 = 0,$$

which is of the form $\quad r^2 - 2r(\alpha \cos \theta - \beta) + (\alpha - \beta)^2 = 0.$

Hence, by (10), the difference between two corresponding elementary arcs is

$$4\sqrt{\alpha\beta} \cos \frac{\theta}{2} d\theta.$$

Consequently, if θ_1 and θ_2 be the values of θ for the two transversals in question, we get the difference of the corresponding arcs

$$= 8\sqrt{\alpha\beta} \left(\sin \frac{\theta_2}{2} - \sin \frac{\theta_1}{2} \right).$$

Also, it can be readily seen that the distance between the vertices of the limaçon is $4\sqrt{\alpha\beta}$; $\therefore$ &c.

18. Show that the length of an arc of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is represented by the integral

$$a^2 b^2 \int \frac{d\theta}{(a^2 \cos^2\theta + b^2 \sin^2\theta)^{\frac{3}{2}}}.$$

This result is easily seen, for we have $ds = \rho d\theta$, and $\rho = \dfrac{a^2 b^2}{p^3}$, $\therefore$ &c.

19. Show, in like manner, that the length of a hyperbolic arc is represented by

$$a^2 b^2 \int \frac{d\theta}{(a^2 \cos^2\theta - b^2 \sin^2\theta)^{\frac{3}{2}}}.$$

20. Hence, prove that the integral

$$\int \frac{dx}{(a - bx^2)^{\frac{3}{2}} (a' - b'x^2)^{\frac{1}{2}}}$$

is represented by an elliptic arc when $ab' > ba'$, and by a hyperbolic arc when $ab' < ba'$.

21. Prove that the differential of the arc of the curve found by cutting in the ratio $n : 1$ the normals to the cycloid

$$y = a + b \cos u, \quad x = au + b \sin u, \text{ is}$$

$$\sqrt{(a + nb)^2 + 4nab \sin^2 \frac{u}{2}}\, du.$$

22. Each element of the periphery of an ellipse is divided by the diameter parallel to it, find the sum of all the elementary quotients extended to the entire ellipse. *Ans.* π.

22. In the figure of page 208, if $\alpha = \angle ACN'$, and $\beta = \angle BCN$, prove that

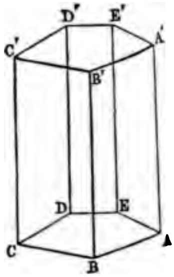
$$\frac{\tan \alpha}{a} = \frac{\tan \beta}{b}.$$

CHAPTER IX.

VOLUMES AND SURFACES OF SOLIDS.

165. **Solids.—The Prism and Cylinder.**—The most simple solid is the cube, which is accordingly the measure of all solids, as the square is that of all areas. Hence the finding the volume of a solid is called its *cubature*. Before proceeding to the application of the Integral Calculus to finding the volumes and surfaces of solids, we propose to show how, in certain cases, such volumes and surfaces can be found from geometrical considerations. In the first place, the volume of a rectangular parallelepiped is measured by the continued product of three adjacent edges; and that of any parallelepiped by the area of a face multiplied by its distance from the opposite face.

Again, the volume of a right prism is measured by the product of its altitude into the area of its base. For example, the volume of the right prism represented in the figure is measured by the area of the polygon $ABCDE$, multiplied by the altitude AA'. Again, since each lateral face, $AB\,B'A'$ for example, is a rectangle, it follows that the sum of the areas of all the faces (exclusive of the two bases), i.e. the area of the surface of the prism, is equal to the rectangle under the altitude and the perimeter of the polygon which forms its base.



This and the preceding result still hold in the limit, when the base, instead of a polygon, is a closed curve of any form, in which case the surface generated is called a *cylinder*. Hence, if V denote the volume of the portion of a cylinder bounded by two planes drawn perpendicular to its edges, h its height, and A the area of its base, we get $V = Ah$.

Again, if Σ denote the superficial area of a cylinder, bounded as before, and S the length of the curve which forms its base, we have $\Sigma = Sh$.

166. **The Pyramid and Cone.**—If the angular points of a polygon be joined to any external point, the solid so formed is called a *pyramid.* Any section of a pyramid by a plane parallel to its base, is a polygon similar to that which forms the base, and the ratio of their homologous sides is the same as that of the distances of the planes from the vertex of the pyramid. Hence it follows that pyramids standing on the same base, and whose vertices lie in a plane parallel to the base, are equal in volume. For, the sections made by any plane parallel to the base are equal in every respect; and, consequently, if we suppose the pyramids divided into an indefinite number of slices by planes parallel to the base, the volumes of the corresponding slices will be the same for all the pyramids; and hence the entire volumes are equal.

Also, if two pyramids have equal altitudes, but stand on different polygonal bases, the volumes of the pyramids will be to each other in the same proportions as the areas of the polygonal bases. For, this proportion holds between the areas of the sections made by any plane parallel to the base; and consequently between the slices made by two infinitely near planes.

Again, the pyramid whose base is one of the faces of a cube, and whose vertex is at the centre of the cube, is the one-sixth part of the cube; for the entire cube can be divided into six equal pyramids, one for each face. Hence, denoting the side of a cube by a, the volume of the pyramid in question is represented by $\frac{a^3}{6}$; i. e. by the product of the area of its base into one-third of its height.

Now, if we vary the base, without altering the height, from what has been established above it follows that the volume of any pyramid is the area of its base into one-third of its height.*

---

* This demonstration is taken from Clairault's "Élémens de Géométrie." The student is supposed familiar with the more ancient proof, from the property that a triangular prism can be divided into three pyramids of equal volume.

If the base of the pyramid be any closed curve, the solid so formed is called a cone; and we infer that the *volume of a cone is equal to one-third of the product of the area of its base into its height.*

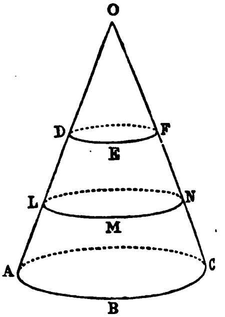
If the base of a pyramid be a regular polygon, and the vertex be equidistant from the angular points of the polygon, the pyramid is called a *right pyramid.*

In this case each *face* of the pyramid is an isosceles triangle, whose area is the rectangle under the side of the polygon and half the perpendicular of the triangle. Hence the surface of the pyramid is equal to the rectangle under the semi-perimeter of the regular polygon and the perpendicular common to each face of the pyramid.

Again, if we suppose the number of sides of the regular polygon to become infinite, the pyramid becomes a right cone; and we infer that the entire surface of a right cone is equal to the rectangle under the semi-circumference of its circular base and the length of an edge of the cone.

Hence, if a be the semi-angle of the cone, l the length of an edge, and r the radius of its base, we have $r = l \sin a$, and the surface of the cone is represented by $\pi l^2 \sin a$.

If a right cone be divided by two planes ABC, DEF, perpendicular to its axis, as in figure, the part intercepted by the planes is called a *truncated* cone.

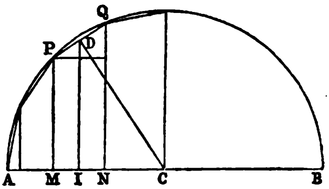


The surface of a truncated cone is easily expressed; for if $OA = l$, $OD = l'$, the required surface is $\pi \sin a\ (l^2 - l'^2)$, or $\pi (l - l')\ (l + l')\ \sin a$.

Now, if the circular section LMN be drawn bisecting the distance between ABC and DEF, the circumference of the circle LMN is $\pi (l + l') \sin a$. Hence the surface of the truncated cone is equal to the rectangle under the edge AD and the circumference of LMN its mean section.

167 **Surface and Volume of a Sphere.**—To find the superficial area of a sphere; suppose a regular polygon inscribed in a semicircle, and let the figure revolve around the diameter AB; then each side of the polygon, PQ for example, will describe a truncated cone.

Now, from the centre C draw CD perpendicular to PQ, and construct as in figure; then, by the preceding article, the surface generated by PQ is equal to $2\pi\ PQ \, . \, DI$.



Again, by similar triangles, we have $DC : DI = PQ : MN$, $\therefore PQ \, . \, DI = DC \, . \, MN$.

Accordingly, since the perpendicular CD is of same length for each side of the polygon, the surface generated by the entire polygon in a complete revolution is equal to $2\pi\ CD \, . \, AB = 4\pi R^2 \cos \dfrac{\pi}{n}$; where n represents the number of sides of the polygon, and R the radius of the circle.

If we suppose n to become infinite the solid generated by the polygon becomes a sphere; and we get $4\pi R^2$ for the entire surface of the sphere. Hence, the surface of a sphere is equal to four times the area of one of its great circles.

Again, it is easy to find the surface generated by any number of sides of the polygon. Thus, for example, that generated by all the sides lying between the points A and Q is plainly equal to $2\pi\ CD \, . \, AN$.

Hence, in the limit, the surface generated in a complete revolution by the arc AQ is equal to $2\pi \, . \, AC \, . \, AN$. Such a portion of a sphere is called a *spherical cap*.

The volume of a sphere is readily found from its surface; for we may regard the volume as consisting of an infinitely great number of pyramids, having their common vertex at the centre, and whose bases form the entire surface. But the volume of each pyramid is represented by the product of one-third of its height (i. e. the radius) by its base. Hence the entire volume of the sphere is one-third of its radius multiplied by its surface, i. e. $\dfrac{4\pi}{3} R^3$.

168. **Surfaces of Revolution.**—In the preceding we have regarded a sphere as generated by the revolution of a circle around a diameter. In general, if any plane be supposed to revolve around a fixed line situated in it, every point in the plane will describe a circle, and any curve lying in the plane will generate a surface.

Such a surface is called a *surface of revolution;* and the fixed line, round which the revolution has place, is called the *axis* of revolution.

It is obvious that the section of a surface of revolution made by any plane drawn perpendicular to its axis is a circle.

If we suppose any solid of revolution to be cut by a series of planes perpendicular to its axis, the volume of the solid intercepted between any two such sections may be regarded as the limit of the sum of an indefinite number of thin cylindrical plates.

Now, if we suppose the generating curve to be referred to rectangular axes, the axis of revolution being that of x, the area of the circle generated by a point (x, y) is plainly equal to πy^2, and the cylindrical plate standing on it, whose thickness is dx, is represented by $\pi y^2 \, dx$.

Hence, the element of volume of the surface of revolution is $\pi y^2 dx$, and the entire volume comprised between two sections, corresponding to the abscissæ α and β, is obviously represented by the definite integral

$$\pi \int_\alpha^\beta y^2 dx,$$

in which the value of y in terms of x is to be got from the equation of the generating curve.

In like manner, the volume of the surface generated by the revolution of a curve around the axis of y is represented by $\pi \int x^2 dy$, taken between suitable limits.

Again, we may regard the surface generated by any element ds of the curve as being ultimately a portion of the surface of a truncated cone, as in Art. 167; and hence the surface generated by ds in a complete revolution round the axis of x is represented by $2\pi y ds$; and accordingly the entire surface generated is represented by

$$2\pi \int y ds,$$

taken between proper limits.

We proceed to apply these formulæ to a few elementary examples.

169. **The Sphere.**—Let $x^2 + y^2 = a^2$ be the equation of the

generating circle; then, substituting $a^2 - x^2$ for y^2, we get for the volume

$$V = \pi \int (a^2 - x^2)\, dx = \pi \left(a^2 x - \frac{x^3}{3} \right) + \text{const.}$$

If we take 0 and a as limits, we get $\frac{2\pi a^3}{3}$ for the volume of the hemisphere; ∴ the entire volume of the sphere is $\frac{4\pi a^3}{3}$, as in Art. 167.

To find the volume of a spherical cap, let h be the length of the portion of the diameter cut off by the bounding plane, and we get for the corresponding volume

$$\pi \int_{a-h}^{a} (a^2 - x^2)\, dx = \pi h^2 \left(a - \frac{h}{3} \right).$$

Again, to find the superficial area, we have

$$ds = \left(1 + \frac{dy^2}{dx^2} \right)^{\frac{1}{2}} dx = \left(1 + \frac{x^2}{y^2} \right) dx = \frac{a}{y}\, dx, \quad \therefore\ y\,ds = a\,dx.$$

Hence, the surface of the zone contained between two parallel planes corresponding to the abscissæ x_1 and x_0 is

$$2\pi \int_{x_0}^{x_1} a\,dx = 2\pi a\,(x_1 - x_0);$$

that is the product of the circumference of a great circle by the breadth of the zone. This agrees with Art. 167.

170. **Right Cone.**—If a denote, as before, the angle which the right line which generates a cone makes with its axis of revolution, we get $y = x \tan a$, taking the vertex of the cone as origin, and the axis of revolution as that of x; accordingly, the element of volume is $\pi \tan^2 a\, x^2 dx$.

Hence, if h denote the height of the cone, we get its volume equal to

$$\pi \tan^2 a \int_0^h x^2\, dx = \frac{\pi}{3} \tan^2 a\, h^3;$$

i. e. $\frac{h}{3}$ × area of its base, as in Art. 166.

Again, to find its surface, we have $ds = \sec\alpha\, dx$,

$$\therefore\ 2\pi \int y\,ds = 2\pi \tan\alpha \sec\alpha \int_0^h x\,dx = \pi h^2 \tan\alpha \sec\alpha;$$

which agrees with the result already obtained.

EXAMPLES.

1. The base of a cylinder is a circle whose area is equal to the surface of a sphere of radius 5 ft.; being given that the volume of the cylinder is equal to the sum of the volumes of two spheres of radii 9 feet and 16 ft., find the height of the cylinder. *Ans.* $64\frac{1}{3}$ ft.

2. A solid sector is cut out of a sphere of 10 ft. radius by a cone, the angle of which is 120°; find the radius of the sphere whose solid contents are equal to those of the sector. *Ans.* $5\sqrt[3]{2}$.

3. Two cones have a common base, the radius of which is 12 ft.; the altitude of one is 9 ft., and that of the other is 5 ft. Find the radius of a sphere whose entire surface is equal to the sum of the areas of the cones. *Ans.* $2\sqrt{21}$ ft.

171. **Paraboloid of Revolution.**—Writing the equation of a parabola in the form $y^2 = 2mx$, we get for the volume of the solid generated by its revolution round the axis of x

$$2\pi m \int x\,dx = \pi m x^2 + \text{const.} = \frac{\pi}{2} y^2 x + \text{const.}$$

Hence, the volume of the surface generated by the revolution of the part of a parabola between its vertex and the point (x_1, y_1) is represented by $\frac{\pi}{2} y_1^2 x_1$, i. e. is equal to half the volume of the circumscribing cylinder.

Again, to find the surface of the paraboloid, we have

$$y\,ds = y\left(1 + \frac{y^2}{m^2}\right)^{\frac{1}{2}} dy = \frac{1}{m}\left(y^2 + m^2\right)^{\frac{1}{2}} y\,dy.$$

Hence, the surface of the paraboloid, between the same limits as above, is represented by

$$\frac{2\pi}{m}\int_0^{y_1} (y^2 + m^2)^{\frac{1}{2}} y\,dy = \frac{2\pi}{3m}\left((y_1^2 + m^2)^{\frac{3}{2}} - m^3\right).$$

172. **Spheroids of Revolution.**— If we suppose an ellipse to revolve round its axis major the surface generated by the

evolving curve is called a *prolate spheroid.* If it revolve ound the axis minor, the surface is called an *oblate spheroid.*

The volume of a spheroid is easily obtained; for, taking $+ \frac{y^2}{b^2} = 1$ as the equation of the curve, we get, on substitut-
ıg $b^2\left(1 - \frac{x^2}{a^2}\right)$ for y^2,

$$V = \pi \frac{b^2}{a^2}\int (a^2 - x^2)\, dx = \frac{\pi b^2}{a^2}\, x \left(a^2 - \frac{x^2}{3}\right) + \text{const.}$$

Ience the entire volume is $\frac{4\pi}{3}ab^2$. In like manner, the vo-
ıme of an oblate spheroid is obviously $\frac{4\pi}{3}ba^2$.

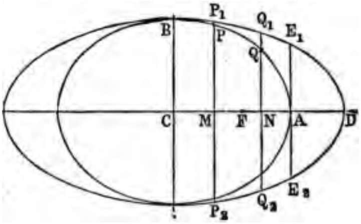
173. **Surface of Spheroid.**—In the case of a prolate sphe-
ɔid we have

$$ds = \left(1 + \frac{b^4x^2}{a^4y^2}\right)^{\frac{1}{2}} dx,$$

$$\cdot\, yds = \left(y^2 + \frac{b^4}{a^4}x^2\right)^{\frac{1}{2}} dx = \left(b^2 - \frac{b^2}{a^2}e^2x^2\right)^{\frac{1}{2}} dx = \frac{be}{a}\left(\frac{a^2}{e^2} - x^2\right)^{\frac{1}{2}} dx.$$

Hence, if $CN = x_1$, $M = x_0$, we get for S, the one generated in a com-
lete revolution by the arc Q,

$$= 2\pi \frac{be}{a}\int_{x_0}^{x_1}\left(\frac{a^2}{e^2} - x^2\right)^{\frac{1}{2}} dx.$$



Iow, if we take $CD = \frac{a}{e}$, and construct an ellipse whose emiaxes are CD and CB, it is easily seen (Art. 126) that he elementary area between two consecutive ordinates of this ellipse is $\frac{be}{a}\left(\frac{a^2}{e^2} - x^2\right)^{\frac{1}{2}} dx$. Hence it follows that the rea of the zone generated by the arc PQ is π times the area of the portion $P_1\, Q_1\, Q_2\, P_2$ of this ellipse.

Again, if AE_1 be the tangent at the vertex of the original ellipse, we see that the entire surface of the spheroid is $4\pi \times$ the area $BCAE_1$; but this is seen, without difficulty, to be

$$2\pi b^2 + 2\pi \frac{ab}{e} \sin^{-1} e. \tag{1}$$

In like manner, we get for the surface S generated by the revolution of an ellipse round its minor axis

$$S = 2\pi \int x ds = 2\pi \int \left(a^2 + \frac{a^4 e^2}{b^4} y^2\right)^{\frac{1}{2}} dy$$

$$= 2\pi \frac{a^2 e}{b^2} \int \left(y^2 + \frac{b^4}{a^2 e^2}\right)^{\frac{1}{2}} dy.$$

If this be integrated, as in Art. 148, we get, after some obvious reductions,

$$S = \pi \frac{ay}{b^2}\left(a^2e^2y^2 + b^4\right)^{\frac{1}{2}} + \pi \frac{b^2}{e} \log \frac{aey + (a^2e^2y^2 + b^4)^{\frac{1}{2}}}{b^2}.$$

If this be taken between the limits 0 and b, and doubled, we get for the entire surface of the ellipsoid

$$2\pi a^2 + \pi \frac{b^2}{e} \log \left(\frac{1 + e}{1 - e}\right). \tag{2}$$

It is readily seen, as in the former case, that the surface of any zone of this ellipsoid is π times the area of a corresponding portion of the hyperbola

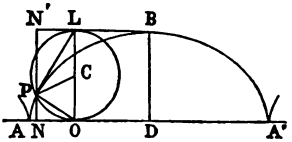
$$\frac{x^2}{a^2} - \frac{a^4 e^2 y^2}{b^4} = 1$$

bounded by lines drawn parallel to the axis of x.

The area of the surface generated by the revolution of a hyperbola round either axis admits of a similar investigation.

Examples.

1. Find the volume of the surface generated by the revolution of a cycloid round its base.



Here, referring the cycloid to *DA* and *DB* as co-ordinate axes, we have (see Dif. Cal., Art. 263)

$$x = a(\phi + \sin\phi), \quad y = a(1 + \cos\phi),$$

where $\angle PCL = \phi$.

Hence

$$dV = \pi y^2\, dx = \pi a^3 (1 + \cos\phi)^3\, d\phi;$$

$\therefore$ for the entire volume V, we get

$$V = 2\pi a^3 \int_0^\pi (1 + \cos\phi)^3\, d\phi = 16\pi a^3 \int_0^\pi \cos^6 \frac{\phi}{2}\, d\phi$$

$$= 32\pi a^3 \int_0^{\frac{\pi}{2}} \cos^6\theta\, d\theta, \quad \text{making } \frac{\phi}{2} = \theta;$$

Hence $$V = 5\pi^2 a^3.$$

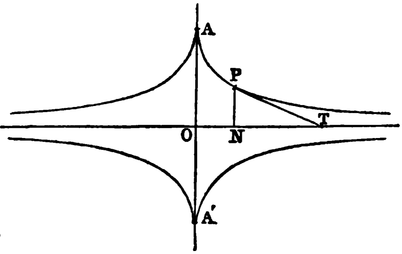
2. Find the whole surface generated in the same case.

Here $$S = 2\pi \int y ds = 4\pi a^2 \int (1 + \cos\phi) \cos\frac{\phi}{2} d\phi:$$

hence the entire surface is

$$32\pi a^2 \int_0^\pi \cos^3 \frac{\phi}{2}\, d\phi = \frac{64\pi a^2}{3}.$$

3. Find the volume and the surface of the solid generated by the revolution of the tractrix round its axis.



(1) Here we have

$$y^2\, dx = -(a^2 - y^2)^{\frac{1}{2}}\, y\, dy;$$

hence the volume generated by the portion *AP* is

$$\pi \int_y^a (a^2 - y^2)^{\frac{1}{2}}\, y\, dy = \frac{\pi}{3} (a^2 - y^2)^{\frac{3}{2}}.$$

The volume generated by the entire tractrix is $\frac{2\pi}{3} a^3$; i.e. half the volume of the sphere whose radius is *OA*.

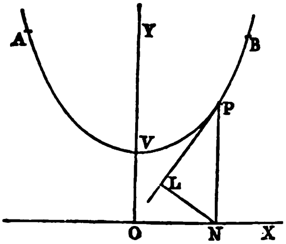
[2.) The surface generated by AP is

$$2\pi\int y ds = 2\pi a\int_y^a dy; \qquad \text{(see page 204),}$$

$$= 2\pi a\,(a - y).$$

Hence the entire surface generated is $2\pi a^2$; i. e. half the surface of the sphere of radius OA.

4. Find the volume, and also the surface, generated by the revolution of the catenary around the axis of x.



(1) here the volume of the solid generated by VP is represented by

$$\pi\int_0^x y^2\,dx = \frac{\pi a^2}{4}\int_0^x \left(e^{\frac{2x}{a}} + e^{\frac{-2x}{a}} + 2\right) dx$$

$$= \frac{\pi a^2}{4}\left\{\frac{a}{2}\left(e^{\frac{2x}{a}} - e^{\frac{-2x}{a}}\right) + 2x\right\}$$

$$= \frac{\pi a}{2}\,(ys + ax):$$

where $s = PV$.

(2). Again, since

$$ds = \tfrac{1}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)dx = \frac{ydx}{a},$$

we have

$$2\pi\int yds = \frac{2\pi}{a}\int y^2 dx.$$

Consequently the surface generated by PV in a complete revolution is $\frac{2}{a}$ × the volume generated; i. e $= \pi\,(ys + ax)$.

5. In the same curve to find the surface generated by its revolution around the axis OV.

Here

$$S = 2\pi\int x\,ds = \pi\int x e^{\frac{x}{a}}\,dx + \pi\int x e^{\frac{-x}{a}}\,dx.$$

Again,

$$\int_0^x x e^{\frac{x}{a}}\,dx = axe^{\frac{x}{a}} - a\int_0^x e^{\frac{x}{a}}\,dx = a\left(x e^{\frac{x}{a}} - a e^{\frac{x}{a}} + a\right).$$

Also

$$\int_0^x x e^{\frac{-x}{a}}\,dx$$

is obtained by ch[illegible]ign of a in the last result.

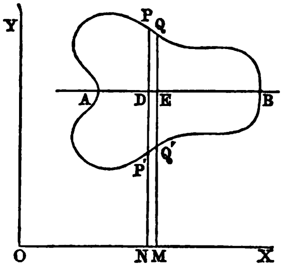
Hence

$$\int_0^x x e^{\frac{-x}{a}} dx = a^2 - axe^{\frac{-x}{a}} - a^2 e^{\frac{-x}{a}}.$$

$$\therefore\ S = \pi \left\{ 2a^2 + ax\left(e^{\frac{x}{a}} - e^{\frac{-x}{a}}\right) - a^2\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)\right\}$$

$$= 2\pi (a^2 + xs - ay).$$

174. If a closed curve, which is symmetrical with respect to a right line, be made to revolve round a parallel line, then the superficial area generated in a complete revolution s equal to the product of the length of the moving curve into the circumference of the circle whose radius is the distance between the parallel lines.



This is easily proved, for let *APBP′* be any curve, symmetrical with respect to *AB*, and suppose *OX* to be the axis of revolution; and draw *PN*, *QM* two infinitely near lines perpendicular to the axis. It is evident that $PQ = P'Q'$. Again let $PN = y$, $P'N = y'$, $PQ = P'Q' = ds$, $DN = b$; then he sum of the elementary zones described by *PQ* and *P′Q′* in a complete revolution is

$$2\pi (y + y')\, ds = 4\pi b\, ds.$$

Consequently the surface generated by the entire curve is πbS, where *S* denotes the whole length of the curve.

A similar theorem holds for the volume of the solid generated: viz., the volume generated is equal to the product f the area of the revolving curve into the circumference of he same circle as before.

For the volume of this solid is plainly represented by

$$\pi \int (y^2 - y'^2)\, dx,$$

r by

$$\pi \int (y - y')\ (y + y')\, dx = 2\pi b \int (y - y')\, dx.$$

ut the area of the curve is represented by

$$\int (y - y')\, dx:$$

consequently; denoting this area by A, and the volume by V, we have

$$V = 2\pi b \times A.$$

In these results the axis of revolution is supposed not to intersect the curve; if it does so, the expression $2\pi b \times A$ represents the difference between the volumes of the surfaces generated by the portions of the curve lying at opposite sides of the axis of revolution; as is readily seen. A similar alteration must be made in the former theorem in this case.

If a circle revolve round any external axis situated in its plane the surface generated is called a spherical ring. From the preceding it follows that the entire surface of such a ring is $4\pi^2 a b$; where a is the radius of the circle, and b the distance of its centre from the axis of revolution.

In like manner the volume of the ring is $2\pi^2 a^2 b$.

175. **Guldin's Theorems.**—The results established in the preceding Article are but particular cases of two general propositions, usually called Guldin's Theorems, but originally enunciated by Pappus (see Walton's Mechanical Problems, p. 39). They may be stated as follows:—

(1). If a plane curve revolve round any external axis, situated in its plane, the area of the surface generated is equal to the product of the perimeter of the revolving curve by the length of the path described, during the revolution, by the centre of gravity of that perimeter.

(2). Under the same circumstances, the volume of the solid generated is equal to the product of the area of the generating curve into the path described by the centre of gravity of the revolving area.

To prove the former, let S denote the whole length of the curve, x, y, the co-ordinates of one of its points, $\bar{x}$, $\bar{y}$, the co-ordinates of the centre of gravity of the curve; then, as is shown in Elementary Mechanics, we have

$$\bar{y} = \frac{\int y\, ds}{S};$$

$$\therefore \quad 2\pi \bar{y}\, S = 2\pi \int y\, ds,$$

i. e, the surface generated by revolution round the axis of x is equal to the product of S, the length of the generating curve, into $2\pi\, y$, the path described by the centre of gravity.

To prove the second proposition; let A denote the area of the generating curve, and dA the element of area corresponding to any point x, y. Also let $\bar{x}, \bar{y}$ be the co-ordinates of the centre of gravity of the area, then

$$\bar{y} = \frac{\Sigma\, y dA}{A} = \frac{\iint y\, dx\, dy}{A} \text{ (substituting } dx\, dy \text{ for } dA),$$

$$\therefore\quad 2\pi\, \bar{y} A = 2\pi \iint y\, dx dy = \pi \int y^2\, dx;$$

where the integral is supposed taken for every point round the perimeter of the curve: but, from Art. 168, the integral at the right-hand side represents the volume of the solid generated; hence the proposition in question follows.

For example, the volume of the ring generated by the revolution of an ellipse around any exterior line situated in its plane, is plainly $2\pi^2 abc$, where a and b are the semiaxes of the ellipse, and c is the distance of its centre from the axis of revolution.

It may be noted that these results still hold if we suppose the curve, instead of making a complete revolution, to turn round the axis through any angle. For, let θ be the circular measure of the angle of rotation, and in the former case we have

$$\theta \bar{y} S = \theta \int y ds.$$

But $\theta \bar{y}$ is the length of the path described by the centre of gravity, and $\theta \int y ds$ is the area of the surface generated by the curve, ∴ &c.

In like manner the second proposition can be shown to hold.

Again, Guldin's theorems are still true if we suppose the rotation to take place around a number of different axes in succession; in which case the centre of gravity, instead of describing a single circle, would describe a number of arcs of circles consecutively; and the whole area of the surface generated will still be measured by the product of the length of the generating curve into the path of its centre of gravity; for this result holds for the part of the surface corresponding to each axis of revolution separately, and therefore holds for the sum.

Again, in the limit, when we suppose each separate rotation indefinitely small, we deduce the following theorem. If any plane curve move so that the path of its centre of gravity is at each instant perpendicular to the moving plane, then the surface generated by the curve is equal to the length of the curve into the path described by its centre of gravity.

The corresponding theorem holds for the volume of the surface generated.

176. The method given in Art. 168 of investigating the volume bounded by a surface of revolution can be readily extended to a solid bounded in any manner. For, if we suppose the volume divided into slices by a system of parallel planes, the entire volume may, as before, be regarded as the limit of the sum of a number of infinitely thin cylindrical plates. Thus, if we suppose a system of rectangular co-ordinate axes taken, and the cutting planes drawn parallel to that of xy; then, if A_z represent the area of the section made by a plane drawn at the distance z from the origin, the entire volume is denoted by

$$\int A_z dz,$$

taken between proper limits.

The area A_z is to be determined in each case as a function of z from the conditions of the bounding surface.

For example, to find the volume of the portion of a cone cut off by any plane; we take the origin at the vertex, and the axis of z perpendicular to the cutting plane; then, if B denote the area of the base, and h the height of the cone, it is easily seen that we have

$$A_z : B = z^2 : h^2, \text{ or } A_z = \frac{Bz^2}{h^2},$$

$$\therefore \quad V = \frac{B}{h^2}\int_0^h z^2\,dz = \tfrac{1}{3}\,B \times h; \text{ as in Art. 166.}$$

If the cutting planes be parallel to that of yz, the volume is denoted by $\int A_x dx$; where A_x denotes the area of the section at the distance x from the origin.

177. **Volume of Elliptic Paraboloid.**—Let it be proposed to find the volume of the portion of the elliptic paraboloid

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z,$$

cut off by a plane drawn perpendicular to the axis of the surface. Here, considering z as constant, the area of the ellipse $\frac{x^2}{p} + \frac{y^2}{q} = 2z$, by Art. 126, is $2\pi z\sqrt{pq}$.

Hence, denoting by c the distance of the bounding plane from the vertex of the surface, we have

$$V = 2\pi\sqrt{pq}\int_0^c z\,dz = \pi\sqrt{pq}\,c^2.$$

This result admits of being exhibited in another form; for if B be the area of the elliptic section made by the bounding plane, we have

$$B = 2\pi c\sqrt{pq}.$$

Hence $V = \frac{1}{2}$ circumscribing cylinder, as in paraboloid of revolution.

178. **The Ellipsoid.**—Next, to find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The section of the surface at the distance z from the origin is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2};$$

the area of this ellipse is

$$\pi\left(1 - \frac{z^2}{c^2}\right)ab, \text{ i.e. } A_z = \pi\left(1 - \frac{z^2}{c^2}\right)a.$$

Hence, denoting the entire volume by V, we have

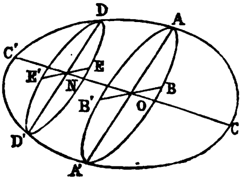
$$V = 2\pi ab\int_0^c\left(1 - \frac{z^2}{c^2}\right)dz = \frac{4}{3}\pi abc.$$

179. It is sometimes more convenient to refer the surface to a system of oblique axes. In this case, if, as before, we take the cutting planes parallel to that of xy, and if ω be the angle the axis of z makes with the plane of xy, the expression for the volume becomes

$$\sin \omega \int A_z \, dz,$$

taken between proper limits, where A_z represents the area of the section, as in the former case.

For example, let us seek the volume of the portion of an ellipsoid cut off by any plane.



Suppose $DED'E'$ to represent the section made by the plane, and $ABA'B'$ the parallel central section. Take OA, OB, the axes of this section, as axes of x and y respectively; and the conjugate diameter OC as axis of z.

Then the equation of the surface is

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1;$$

where $OA = a'$, $OB = b'$, $OC = c'$.

It will now be convenient to transfer the origin to the point C', without altering the directions of the axes, when the equation of the surface becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = \frac{2z}{c'} - \frac{z^2}{c'^2}.$$

The area A_z of the section, by Art. 126, is

$$\pi a'b' \left(\frac{2z}{c'} - \frac{z^2}{c'^2}\right);$$

hence, denoting $C'N$ by h, the volume cut off by the plane DED' is represented by

$$\pi a'b' \sin \omega \int_0^h \left(\frac{2z}{c'} - \frac{z^2}{c'^2}\right) dz,$$

or

$$\pi a'b' \sin \omega \left(\frac{h^2}{c'} - \frac{h^3}{3c'^2}\right).$$

But, by a well-known theorem,* we have

$$a'b'c' \sin \omega = abc,$$

where a, b, c, are the principal semiaxes of the surface.

Hence the expression for the volume V in question becomes

$$V = \pi abc\left(\frac{h^2}{c'^2} - \frac{h^3}{3c'^3}\right); \tag{3}$$

or, denoting $\frac{C'N}{C'O}$ by k,

$$V = \pi abck^2\left(1 - \frac{k}{3}\right). \tag{4}$$

This result shows that the volume cut off is constant for all sections for which k has the same value. Again, since $\frac{ON}{OC'} = 1 - k$, the locus of N is a similar ellipsoid; and we infer that *if a plane cut a constant volume from an ellipsoid the locus of the centre of the section is a similar and similarly situated ellipsoid.*

The corresponding results for the elliptic paraboloid can be deduced from the preceding by adopting the usual method of such derivation: viz., by taking

$$a^2 = pc, \quad b^2 = qc,$$

and afterwards making c infinite: observing that in this case the ratio $\frac{c}{c'}$ becomes unity.

Making these substitutions in (3) it becomes†

$$V = \pi\sqrt{pq}\,h^2\left(1 - \frac{h}{3c'}\right), \text{ or } \pi h^2\sqrt{pq}, \text{ since } c' = \infty.$$

Hence, if a constant length be measured on any diameter of an elliptic paraboloid and a conjugate plane drawn, then the volume† of the segment cut from the paraboloid by the plane is constant.

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* Salmon's Geometry of Three Dimensions, Art. 96.

† For a more direct investigation the student is referred to a memoir "On some Properties of the Paraboloid," Quarterly Journal of Mathematics, June, 1874, by Professor Allman.

EXAMPLES.

1. A cylinder intersects the plane of xy in an ellipse of semiaxes $OA = a$, $OB = b$, and the plane of xz in an ellipse of semiaxes $OA = a$, $OC = c$; the edges of the cylinder being parallel to BC; find the volume of the portion of the cylinder bounded by the three co-ordinate planes. *Ans.* $\frac{1}{3} abc$.

2. The axes of two equal right cylinders intersect at right angles; find the volume common to both. *Ans.* $\frac{16}{3} a^3$, where a is the radius of either cylinder. This surface is called a *Groin*.

180. **Volume by Double Integration.**—In the application of the preceding method of finding volumes, the area represented by A_x, instead of being immediately known, requires in general a previous integration; so that the determination of the volume of a surface involves two successive integrations, and consequently V is expressed by a *double* integral.

Thus, as the area A_x lies in a plane parallel to that of yz, its value, as in Art. 124, may generally be represented by $\int z dy$, taken between proper limits. Hence V may be represented by

$$\int \left[\int z dy \right] dx,$$

or, adopting the usual notation, by

$$\iint z dy dx,$$

taken between limits determined by the data of the question.

The value of z is supposed given by a relation $z = f(x, y)$, by means of the equation of the bounding surface; hence

$$\int z dy = \int f(x, y)\, dy.$$

In the determination of this integral, we regard x as a constant (since all the points in the area have the same value of x), and integrate with respect to y between its proper limits.

Thus, if y_1 and y_0 denote the limiting values of y, the definite integral

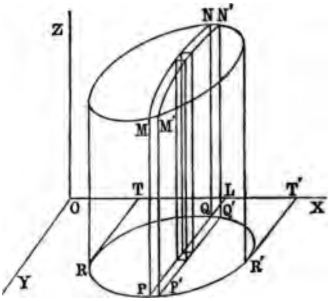
$$\int_{y_0}^{y_1} f(x, y)\, dy$$

becomes a function of x; this function, when integrated with respect to x between the proper limits, determines the volume in question.

If x_1 and x_0 denote the limits of x, V may be represented by the double integral

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x, y)\,dydx.$$

We shall exemplify this by a figure, in which we suppose the volume bounded by the plane of xy, by a cylinder perpendicular to that plane, and also by any surface.* Let $RPR'Q$ represent the section of the cylinder by the plane of xy; and suppose $PMNQ$ to be the section of the volume by a parallel to yz at the distance x from the origin. Let $PL = y_1$, $QL = y_0$, then the area $PMNQ$ is represented by the integral



$$\int_{y_0}^{y_1} z dy.$$

The values† of y_1 and y_0 in terms of x are obtained from the equation of the curve $RPR'Q$.

Again, suppose $P'M'N'Q'$ to represent the parallel section at the infinitesimal distance dx from $PMNQ$, then the elementary plate between $PMNQ$ and $P'M'N'Q'$ is represented by

$$dx\int_{y_0}^{y_1} z dy.$$

Now, if RT and $R'T'$ be tangents to the bounding curve, perpendicular to the axis of x, and if $OT' = x_1$, $OT = x_0$, the entire volume is represented by

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* The determination of a volume of any form is virtually contained in this. For, if we suppose the surface circumscribed by a cylinder perpendicular to the plane of xy, the required volume will become the difference between two cylinders, bounded by the upper and lower portions of the surface, respectively. See Bertrand, Cal. Int. § 447.

† In our investigation we have assumed that the parallels intersect the curve in but two points each; the general case is omitted, as the solution in such cases can be rarely obtained, and also as the investigation is unsuited for an elementary treatise.

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} z\,dy\,dx.$$

It should be observed that $z\,dy\,dx$ represents the volume of the parallelopiped whose height is z, and whose base is the infinitesimal rectangle having dx and dy as sides; and consequently the volume may be regarded as the sum of all such parallelopipeds corresponding to every point within the area $RPR'Q$.

It is also plain that we shall arrive at the same result whether we integrate first with respect to x; and afterwards with respect to y, or *vice-versâ;* i. e. whether we conceive the volume divided into slices parallel to the plane of xz, or to that of yz.

We shall illustrate the preceding by an example.*

Suppose $RPR'Q$ to be the circle

$$(x - a)^2 + (y - b)^2 = R^2,$$

and the bounding surface the hyperbolic paraboloid

$$xy = cz;$$

then we have

$$y_0 = b - \sqrt{R^2 - (x - a)^2}, \quad y_1 = b + \sqrt{R^2 - (x - a)^2},$$

and

$$\int_{y_0}^{y_1} z dy = \frac{1}{c}\int_{y_0}^{y_1} xy dy = \frac{x}{2c}(y_1^2 - y_0^2) = \frac{2bx}{c}\sqrt{R^2 - (x - a)^2}.$$

Again $\quad x_1 = a + R, \quad x_0 = a - R,$

$$\therefore \; V = \frac{2b}{c}\int_{a-R}^{a+R}\sqrt{R^2 - (x - a)^2}\,x dx.$$

Now let $x - a = R\sin\theta$, and we get

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* This example is taken from Cauchy's Applications Géométriques du Calcul Infinitesimal, p. 109.

$$V = \frac{2bR^2}{c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,(a + R\sin\theta)\,d\theta.$$

$$\text{But } \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{\pi}{2},\quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\sin\theta\,d\theta = 0,$$

$$\therefore\; V = \pi\frac{abR^2}{c}.$$

181. **Double Integration.**—From the preceding Article it is readily seen that the double integral

$$\iint f(x, y)\,dy\,dx$$

can be represented geometrically by a volume; and the determination of the double integral, when the limits are given, is the same as the finding the volume of a solid with corresponding limits.

For instance, the example in the preceding page is equivalent to finding the value of the double integral

$$\iint xy\,dx\,dy$$

taken for all values of x and y subject to the condition

$$(x - a)^2 + (y - b)^2 - R^2 < 0;$$

and similarly in other cases.

When the limits of x and y are constants, as in

$$\int_a^{a'}\int_b^{b'} f(x, y)\,dy\,dx,$$

the double integral represents the volume cut by the surface

$$z = f(x, y)$$

from the parallelopiped whose base is the rectangle formed by the lines

$$x = a,\quad x = a',\quad y = b,\quad y = b'.$$

It is plain that in this case the order of integration is indifferent, as already seen in Art. 113.

182. It is sometimes more convenient to refer the curve $RPR'Q$ to polar co-ordinates, in which case we conceive the area divided into infinitesimal rectangles of the type $rdrd\theta$.

The corresponding parallelopiped is represented by $zrdrd\theta$, and the expression for V becomes

$$V = \iint zr\,dr\,d\theta,$$

taken between proper limits.

For instance, if the bounding surface be a sphere, whose centre is the origin, we have

$$z = \sqrt{a^2 - r^2},$$

and the equation becomes

$$V = \iint \sqrt{a^2 - r^2}\, r\,dr\,d\theta\,;$$

but
$$\int \sqrt{a^2 - r^2}\, r\,dr = -\tfrac{1}{3}(a^2 - r^2)^{\frac{3}{2}}.$$

Hence, if V denote the volume included between the sphere and the *exterior* surface of the cylinder, we shall have

$$V = \tfrac{1}{3}\int (a^2 - r^2)^{\frac{3}{2}}\,d\theta,$$

where we suppose each *radius* of the sphere to cut the cylinder in but one point.

For example, let the base of the cylinder be the pedal of an ellipse whose major axis coincides with a diameter of the sphere; then

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta,$$

and
$$V = \tfrac{1}{3}(a^2 - b^2)^{\frac{3}{2}}\int \sin^3\theta\,d\theta.$$

If this be integrated between the limits 0 and $\frac{\pi}{2}$, we get the $\frac{1}{8}$th of the entire volume; hence

$$V = \frac{16}{9}(a^2 - b^2)^{\frac{3}{2}}.$$

EXAMPLES.

1. A sphere is cut by a right cylinder, the radius of whose base is half that of the sphere, and one of whose edges passes through the centre of the sphere; find the volume common to both surfaces.

Ans. $\frac{2\pi a^3}{3} - \frac{8a^3}{9}$, a being the radius of the sphere.

2. If the base of the cylinder be the complete curve represented by the equation $r = a \cos n\theta$, where n is any integer, find the volume of the solid between the surface of sphere and the external surface of the cylinder.

183. It is readily seen, as in Art. 138, that *the volume included within the surface represented by the equation*

$$F\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) = 0$$

is abc × the volume of the surface

$$F(x, y, z) = 0.$$

For, let $\frac{x}{a} = x'$, $\frac{y}{b} = y'$, $\frac{z}{c} = z'$, and we shall have

$$z\,dx\,dy = abc\,z'\,dx'\,dy',$$

and
$$\therefore \iint z\,dx\,dy = abc \iint z'\,dx'\,dy';$$

which proves the theorem.

Hence, for example, the determination of the volume of an ellipsoid is reduced to that of a sphere.

Again, if the point (x, y, z) move along a plane, the corresponding point (x', y', z') will describe another plane. From this property the expression for the volume of an ellipsoidal cap (Art. 179) can be immediately deduced from that of a spherical cap (Art. 169).

In like manner the volume included between a cone enveloping an ellipsoid and the surface of the ellipsoid is reducible to the corresponding volume for a sphere.

184. **Quadrature on the Sphere.**—We next propose to give a brief discussion of quadrature on a sphere, and commence with the results on the subject usually given in treatises on Spherical Trigonometry. In the first place, since the area of a lune is to that of the entire sphere as the angle of the lune to four right angles, the area of a lune of angle A is represented by $2R^2A$; where R is the radius of the sphere, and A is expressed in circular measure.

Again, the area of a spherical triangle ABC is expressed by $R^2(A + B + C - \pi)$; for, the sum of the three lunes exceeds the hemisphere by twice the area of the triangle, as is easily seen from a figure.

Hence, it readily follows that the area Σ of a spherical polygon of n sides is represented by

$$\Sigma = R^2 \{A + B + C + \&c. - (n-2)\pi\};$$

A, B, C, &c., being the angles of the polygon.

This result admits of being expressed in terms of the sides of the polar polygon; for, representing these sides by a', b', c', &c., we have

$$A = \pi - a', \quad B = \pi - b', \ \&c.,$$

and consequently

$$\Sigma = R^2 \{2\pi - (a' + b' + c' + \&c.)\}.$$

Or, denoting the perimeter of the polar figure by S,

$$\Sigma + RS = 2\pi R^2. \qquad (5)$$

This proof is perfectly general, and holds in the limit, when the polygon becomes any curve; and, accordingly, the area bounded by any closed spherical curve is connected with the perimeter of its polar curve by the relation (5).

Again, the spherical area bounded by a *lesser* circle (Art. 167) admits of a simple expression; for, if ρ denote the circular radius of the circle, or the arc from its pole to its circumference, the area in question is represented by

$$2\pi R^2(1 - \cos\rho);$$

for (see fig. Art. 167) we have

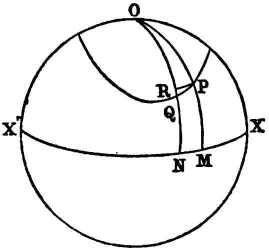
$$AN = AC - CN = R(1 - \cos\rho).$$

This result also follows immediately as a simple case of equation (5).

Again, the area bounded by the lesser circle and by two arcs drawn to its pole is plainly represented by

$$R^2 a(1 - \cos\rho),$$

where a is the angle between the arcs.



We can now find an expression for the area bounded by any closed curve on a sphere; for the position of any point P on the surface can be expressed by means of the arc OP drawn to a fixed point, and of the angle POX between this arc and a fixed arc through O. These are called the polar co-ordinates of the point, and are analogous to ordinary polar co-ordinates on a plane.

Now, let $OP = \rho$, and $POX = \omega$, and any curve on the sphere may be supposed to be expressed by a relation between ρ and ω.

Again, suppose OQ to be an infinitely near vector, and draw PR perpendicular to OP; then, neglecting the area PQR, the elementary area OPQ, by the preceding, is represented by

$$R^2 \int (1 - \cos\rho)\, d\omega.$$

Hence the area bounded by two vectors from O is expressed by the integral $R^2 \int (1 - \cos\rho)\, d\omega$, taken between suitable limits.

If the curve be closed, the entire superficial area becomes

$$R^2 \int_0^{2\pi} (1 - \cos\rho)\, d\omega.$$

The value of $\cos\rho$ in terms of ω is to be determined in each case by means of the equation of the bounding curve.

The integral $R^2 \int_0^{2\pi} \cos\rho \, d\omega$ obviously represents the area included between the closed curve and the great circle which has O for its pole.

The length of the curve can also be represented by a definite integral; for, regarding PRQ as ultimately a right-angled triangle, we have, in the limit,

$$PQ^2 = PR^2 + RQ^2: \text{ also } PR = \sin\rho \, d\omega.$$

Hence
$$ds^2 = d\rho^2 + \sin^2\rho \, d\omega,$$

or
$$ds = d\omega \sqrt{\sin^2\rho + \left(\frac{d\rho}{d\omega}\right)^2}$$

$$\therefore \; s = \int d\omega \sqrt{\sin^2\rho + \left(\frac{d\rho}{d\omega}\right)^2}.$$

EXAMPLES.

1. Find the area of the portion of the surface of a sphere which is intercepted by a right cylinder, one of whose edges passes through the centre of the sphere, and the radius of whose base is half that of the sphere.

Here, the equation of the base may be written in the form $r = R \sin\omega$, R being the radius of the sphere, and ω being measured from the tangent to the circular base.

Again, from the sphere we have $r = R \sin\rho$, $\therefore \rho = \omega$ is the equation of the curve of intersection of the sphere and the cylinder; hence the area in question is

$$2R^2 \int_0^{\frac{\pi}{2}} (1 - \cos\omega) \, d\omega = 2R^2\left(\frac{\pi}{2} - 1\right).$$

This being doubled gives the whole intercepted area $= 2\pi R^2 - 4R^2$.

In general, if $r = f(\omega)$ be the equation of the base of a cylinder, it is easily seen that the equation of the curve of its intersection with the sphere may be written in the form $R \sin\rho = f(\omega)$.

For example, let the diameter of the right cylinder be less than half that of the sphere, and, writing the equation of the base in the form $r = a \sin\omega$, where a is the diameter of the section, we get $R \sin\rho = a \sin\omega$, or $\sin\rho = \kappa \sin\omega$ (where κ is < 1), as the equation of the curve of intersection of the sphere and the cylinder.

Hence the intercepted area is denoted by

$$2R^2 \int_0^{\frac{\pi}{2}} \left(1 - \sqrt{1 - \kappa^2 \sin^2\omega}\right) d\omega = \pi R^2 - 2R^2 \int_0^{\frac{\pi}{2}} \sqrt{1 - \kappa^2 \sin^2\omega} \, d\omega.$$

Hence the area in question depends on the rectification of an ellipse.

2. Find the area of the portion of the surface of the cylinder intercepted by the sphere, in the preceding.

Here the area in question is easily seen to be represented by $2\int z\,ds$, where ds denotes the element of the curve which forms the base, corresponding to the edge z.

Now (1), when the diameter of the base is equal to the radius of the sphere, we have

$$z = R\cos\omega, \text{ and } ds = R\,d\omega,$$

$$\therefore \text{ area in question} = 2R^2\int_0^{\frac{\pi}{2}}\cos\omega\,d\omega = 4R^2.$$

(2). When the diameter is less than the radius of the sphere,

$$2\int z\,ds = 2a\int\sqrt{R^2 - a^2\sin^2\omega}\,d\omega = 2aR\int\sqrt{1 - \kappa^2\sin^2\omega}\,d\omega, \therefore \text{ \&c.}$$

185. **Quadrature of Surfaces.**—In seeking the area of a portion of any surface we regard it as the limit of a number of infinitely small elements, each of which is considered as a portion of a plane which is ultimately a tangent plane to the surface. Now let dS denote such an element of the superficial area, and $d\sigma$ its projection on a fixed plane which makes the angle θ with the plane of the element; then, from elementary geometry, we shall have

$$d\sigma = \cos\theta\,dS, \text{ or } dS = \sec\theta\,d\sigma.$$

Hence $$S = \int\sec\theta\,d\sigma,$$

taken between suitable limits.

The applications of this formula usually involve *double integration*, and are generally very complicated; there is, however, one mode by which the determination of the area of a portion of a surface can be reduced to a single integration, and by whose aid its value can in some cases be found; viz., by supposing the surface divided into zones by a system* of curves along each of which the angle θ between the tangent plane and a fixed plane is constant; then, if dS denote the superficial area of the zone between the two infinitely near curves corresponding to the angles θ and $\theta + d\theta$; and, if dA be the projection of this area on the fixed plane, we plainly have $dS = \sec\theta\, dA$.

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* I am not aware to whom this method is due, but it has been employed in a more or less modified form by M. Catalan, Liouville, tome iv., p. 323, by Mr. Jellett, Camb. and Dub. Math. Journal, vol. i., as also by other writers. Such curves are called *parallel curves* by M. Lebesgue, Liouville, tome xi., p. 332, and "Curven isokliner Normalen," by Dr. Schlömilch.

If we suppose the surface referred to a rectangular system of axes, the fixed plane being that of xy; and, adopting the usual notation, if we take λ, μ, ν as the direction angles of the normal at any point on the surface, we get for dS, the area of the zone between the curves corresponding to ν and $\nu + d\nu$, the equation

$$dS = \sec\nu\, dA,$$

where A denotes the area of the projection on the plane of xy of the closed curve defined by the equation $\nu =$ constant.

Now whenever we can express the area A in terms of ν and constants, then the area of a portion of the surface, bounded by two curves of the system in question, is reducible to a single integration.

The most important applications of this method are furnished by surfaces of the second degree, to which we proceed to apply it, commencing with the paraboloid.

186. **Quadrature of the Paraboloid.**—Writing the equation of the surface in the form

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z,$$

the equation of the tangent plane at the point (x, y, z) is

$$\frac{xX}{p} + \frac{yY}{q} = z + Z,$$

where X, Y, Z are the co-ordinates of any point on the plane.

Comparing this with the equation

$$X\cos\lambda + Y\cos\mu + Z\cos\nu = P,$$

we get $$\cos\lambda = \frac{x}{p}\cos\nu, \quad \cos\mu = \frac{y}{q}\cos\nu;$$

substituting in the identical equation

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1,$$

we get
$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = \tan^2\nu. \qquad (6)$$

Consequently the curve along which the tangent plane makes the angle ν with the tangent plane at the vertex is projected on that plane into the ellipse

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = \tan^2\nu.$$

The area A of this ellipse is $\pi pq \tan^2\nu$; accordingly, we have

$$dA = \pi pq d(\tan^2\nu),$$

$$\therefore\ dS = \pi pq \sec\nu\, d(\tan^2\nu) = \pi pq \sec\nu\, d(\sec^2\nu);$$

hence the area of the paraboloidal cap bounded by the curve $\nu = a$ is

$$\pi pq \int_0^a \sec\nu\, d(\sec^2\nu) = \tfrac{2}{3}\pi pq\,(\sec^3 a - 1).$$

Also the area of the belt* between the curves

$$\nu = a \text{ and } \nu = a' \text{ is } \tfrac{2}{3}\pi pq\,(\sec^3 a' - \sec^3 a).$$

187. **Quadrature of The Ellipsoid.**—Proceeding in like manner to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

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[*] This form for the quadrature of a paraboloid is, I believe, due to Mr. Jellett; see Camb. and Dub. Math. Journal, vol. i., p. 65. The proof given above is in a great measure taken from Mr. Allman's paper in the Quarterly Journal, already referred to.

the equation of the tangent plane at the point (x, y, z) is

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} + \frac{Zz}{c^2} = 1.$$

Hence, comparing with the equation

$$X \cos \lambda + Y \cos \mu + Z \cos \nu = P,$$

we get

$$\cos \lambda = \frac{c^2}{a^2} \frac{x}{z} \cos \nu, \qquad \cos \mu = \frac{c^2}{b^2} \frac{y}{z} \cos \nu.$$

Hence, we have

$$\cos^2 \nu \frac{c^4}{z^2} \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} \right) = \cos^2 \lambda + \cos^2 \mu = \sin^2 \nu;$$

or, substituting $1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$ for $\frac{z^2}{c^2}$,

$$\frac{x^2}{a^4} \left(a^2 \sin^2 \nu + c^2 \cos^2 \nu \right) + \frac{y^2}{b^4} \left(b^2 \sin^2 \nu + c^2 \cos^2 \nu \right) = \sin^2 \nu.$$

This shows that the projection on the plane of xy, of a curve along which ν = constant, is an ellipse.

Again, the area A of this ellipse is

$$\frac{\pi a^2 b^2 \sin^2 \nu}{(a^2 \sin^2 \nu + c^2 \cos^2 \nu)^{\frac{1}{2}} \, (b^2 \sin^2 \nu + c^2 \cos^2 \nu)^{\frac{1}{2}}},$$

and, accordingly, the area dA of the elementary annulus between two consecutive ellipses is

$$\pi a^2 b^2 \frac{d}{d\nu} \left\{ \frac{\sin^2 \nu}{(a^2 \sin^2 \nu + c^2 \cos^2 \nu)^{\frac{1}{2}} \, (b^2 \sin^2 \nu + c^2 \cos^2 \nu)^{\frac{1}{2}}} \right\} d\nu.$$

The corresponding elementary ellipsoidal zone dS is represented by

$$\frac{\pi a^2 b^2}{\cos \nu} \frac{d}{d\nu} \left\{ \frac{\sin^2 \nu}{(a^2 \sin^2 \nu + c^2 \cos^2 \nu)^{\frac{1}{2}} \, (b^2 \sin^2 \nu + c^2 \cos^2 \nu)^{\frac{1}{2}}} \right\} d\nu.$$

Now, if S denote the superficial area between two curves corresponding to $\nu = \alpha$ and $\nu = \alpha'$, after one or two reductions, it is easily seen that

$$S = \pi a^2 b^2 c^2 (I + I'),$$

where $$I = \int_{\alpha}^{\alpha'} \frac{\sin\nu\, d\nu}{(b^2 \sin^2\nu + c^2 \cos^2\nu)^{\frac{3}{2}} (a^2 \sin^2\nu + c^2 \cos^2\nu)^{\frac{1}{2}}},$$

$$I' = \int_{\alpha}^{\alpha'} \frac{\sin\nu\, d\nu}{(a^2 \sin^2\nu + c^2 \cos^2\nu)^{\frac{3}{2}} (b^2 \sin^2\nu + c^2 \cos^2\nu)^{\frac{1}{2}}}.$$

Now, the former of these integrals is represented by an arc of an ellipse, and the latter by an arc of a hyperbola; it being assumed that $a > b > c$.

For, assuming $a^2 - c^2 = a^2 e^2$, and $b^2 - c^2 = b^2 e'^2$, and making $\cos\nu = x$, we get

$$I = \frac{1}{ab^3} \int_{\cos\alpha'}^{\cos\alpha} \frac{dx}{(1 - e'^2 x^2)^{\frac{3}{2}} (1 - e^2 x^2)^{\frac{1}{2}}},$$

$$I' = \frac{1}{a^3 b} \int_{\cos\alpha'}^{\cos\alpha} \frac{dx}{(1 - e^2 x^2)^{\frac{3}{2}} (1 - e'^2 x^2)^{\frac{1}{2}}}.$$

Again, let $ex = \sin\theta$ in the former integral, and $e'x = \sin\theta$ in the latter, and we get

$$I = \frac{e^2}{ab^3} \int \frac{d\theta}{(e^2 - e'^2 \sin^2\theta)^{3}}$$

$$I' = \frac{e'^2}{a^3 b} \int \frac{d\theta}{(e'^2 - e^2 \sin^2\theta)^{\frac{3}{2}}}.$$

Now, since $e > e'$, the former integral represents an arc of an ellipse, and the latter an arc of a hyperbola. (See Ex. 18, p. 225).

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[*] This form for the quadrature of an ellipsoid is given by Mr. Jellett in the memoir already referred to. He has also shown that the ellipse and the hyperbola in question are the focal conics of the reciprocal ellipsoid; a result which can be easily arrived at from the forms of I and I' given above.

For application to the hyperboloid, and further development of these results the student is referred to Mr. Jellett's memoir.

EXAMPLES.

1. A sphere of 15 feet radius is cut by two parallel planes, at distances of 3 and 7 feet from its centre; find the superficial area of the portion of the surface included between the planes. *Ans.* 376.9908 sq. feet.

2. Being given the slant height of a right cone, find the cosine of half its vertical angle when its volume is a maximum. *Ans.* $\dfrac{1}{\sqrt{3}}$.

3. Prove that the volume of a truncated cone of height h is represented by

$$\frac{\pi h}{3}(R^2 + Rr + r^2),$$

where R and r are the radii of its two bases.

4. A cone is circumscribed to a sphere of radius R, the vertex of the cone being at the distance D from the centre; find the ratio of the superficial area of the cone to that of the sphere. *Ans.* $\dfrac{D^2 - R^2}{4DR}$.

5. Two spheres, A and B, have for radii 9 feet and 40 feet; the superficial area of a third sphere C is equal to the sum of the areas of A and B; calculate the excess, in cubic feet, of the volume of C over the sum of the volumes of A and B. *Ans.* 17558.

6. If any arc of a plane curve revolve successively round two parallel axes, show that the difference of the surfaces generated is equal to the product of the length of the arc into the circumference of the circle described by any point on either axis turning round the other.

If the axes of revolution lie at opposite sides of the curve, the sum of the surfaces must be taken instead of the difference.

7. Find, in terms of the sides, the volume of the solid generated by the complete revolution of a triangle round its side c.

Ans. $\dfrac{4\pi}{3}\,\dfrac{s(s-a)(s-b)(s-c)}{c}$.

8. Apply Guldin's theorem to determine the distance from the centre, of the centre of gravity, (1) of a semicircular area; (2) of a semicircular arc.

Ans. (1) $\dfrac{4a}{3\pi}$, (2) $\dfrac{2a}{\pi}$.

9. If a triangle revolve round any external axis, lying in its plane, find an expression for the area of the surface generated in a complete revolution.

10. Prove that the volume cut from the surface

$$z^n = Ax^2 + By^2$$

by any plane parallel to that of xy, is $\dfrac{1}{n+1}$th part of the cylinder standing on the plane section, and terminated by the plane of xy.

11. If a sphere and its circumscribing cylinder be cut by planes perpendicular to the axis of the cylinder, prove that the intercepted portions of the surfaces are equal in area.

12. The axis of a right circular cylinder passes through the centre of a sphere, find the volume of the solid included between the concave surface of the sphere and the convex surface of the cylinder.

Ans. $\frac{\pi c^3}{6}$, where c is the length of the portion of any edge of the cylinder intercepted by the sphere.

This question is the same as that of finding the volume of the solid generated by the segment of a circle cut off by any chord, in a revolution round the diameter parallel to the chord.

13. Find the volume of the solid generated by the revolution of an arc of a circle round its chord. *Ans.* $2\pi a \left\{\frac{(2a^2 + c^2)\sin\alpha}{3} - c\alpha\right\}$,

where a = radius, c = distance of chord from centre, and $\cos\alpha = \frac{c}{a}$.

In this we suppose the arc less than a semicircle; the modification when it is greater is easily seen.

14. If the ellipsoid of revolution

$$x^2 + z^2 + \frac{a^2 y^2}{b^2} = a^2,$$

and the hyperboloid

$$x^2 + z^2 - \frac{a^2 - b^2}{b^2} y^2 = a^2,$$

be cut by two planes perpendicular to the axis of revolution, prove that the zones intercepted on the two surfaces are of equal area.

15. Find the entire volume bounded by the surface

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1, \qquad \textit{Ans. } \frac{abc}{90}.$$

and by the positive side of the three co-ordinate planes.

16. Find the volume of the surface generated by the revolution of an arc of a parabola round its chord; the chord being perpendicular to the axis of the curve.

Ans. $\frac{8}{15}\pi b^2 c$, where c is the length of the chord, and b the intercept made by it on the diameter of the parabola passing through the middle point of the chord.

17. A sphere of radius r is cut by a plane at distance d from the centre; find the difference of the volumes of the two cones having as a common base the circle in which the plane cuts the sphere, and whose vertices are the opposite ends of the diameter perpendicular to the cutting plane.

Ans. $\frac{2}{3}\pi d\,(r^2 - d^2)$.

18. Find the area of a spherical triangle; and prove that if a curve traced on a sphere have for its equation $\sin\lambda = f(l)$, λ denoting latitude, and l longitude, the area between the curve and the equator $= \int f(l)\,dl$.

19. Show that the volume contained between the surface of an hyperboloid of one sheet, its asymptotic cone, and two planes parallel to that of the real axes, is proportional to the distance between those planes.

20. In example of Art. 180 find the entire surface of the portion of the cylinder included between the hyperbolic paraboloid and the plane of xy.

21. Find the entire volume of the surface

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1. \qquad \textit{Ans. } \frac{4\pi abc}{5 \cdot 7}.$$

22. The vertex of a cone of the second degree is in the surface of a sphere, and its internal axis is the diameter passing through its vertex; find the volume of the portion of the spherical surface intercepted within the cone.

23. Prove that the volume of the portion of a cylinder intercepted between any two planes, is equal to the product of the area of a perpendicular section into the distance between the centres of gravity of the areas of the bounding sections.

MISCELLANEOUS EXAMPLES.

1. If a be the sagitta of a circular segment whose base is b, prove that the area of the segment is, approximately,

$$= \frac{2}{3}ab + \frac{a^3}{2b}.$$

2. Find the area of the inverse of a hyperbola, the centre being the pole of inversion; and shew that the area of the inverse of an ellipse, under the same circumstances, is an arithmetic mean between the areas of the circles described on its axes as diameters.

3. Integrate the expression

$$\sqrt{\frac{a\sin^2\theta + b\cos^2\theta}{\sin\theta}}\,d\theta.$$

4. If the symbols sin hyp. A, cos hyp. A, represent the hyperbolic sine and cosine of A (see Ex. 1, p. 172); find the values of sin hyp. $(A+B)$ and cos hyp. $(A+B)$ in terms of the hyperbolic sines and cosines of A and B.

Ans. Cos hyp. $(A+B)$ = cos hyp. A cos hyp. B + sin hyp. A sin hyp. B,
sin hyp. $(A+B)$ = sin hyp. A cos hyp. B + cos hyp. A sin hyp. B.

5. In a spiral of Archimedes, if P, Q, and P', Q' be the points of section with any two branches of the curve made by a line passing through its pole; prove that the area bounded by the right line and by the two branches is half the area of the ellipse whose semi-axes are PP' and $P'Q$.

6. Find the value of

$$\int\frac{dx}{x+a}\sqrt{\frac{x+a}{x+b}}.$$

7. If an ellipse roll upon a right line, show that the differential equation of the locus of its focus is

$$(y^2+b^2)\frac{dy}{dx} = \sqrt{(2ay+y^2+b^2)(2ay-y^2-b^2)}.$$

8. A circle rolls from one end to the other of a curved line equal in length to the circumference of the circle, and then rolls back again on the other side of the curve; prove that, if the curvature of the curve be throughout less than that of the circle, the area contained within the closed curve traced out by the point of the circle which was first in contact with the fixed curve is six times the area of the circle.

9. In the same case show that the entire length of the path described is 8 times the diameter of the circle.

10. Prove that the area between two focal radii of a parabola and the curve is half the area between the curve, the corresponding perpendiculars on the directrix, and the directrix.

11. Evaluate the following integrals:

$$\int\frac{dx}{\sqrt{\tan x}},\quad \int\sqrt{\sec x - 1}\,dx,\quad \int\frac{(1+x)^4\,dx}{(1+x^2)(1+x^4)}.$$

12. Being given any plane closed curve; if ds represent an infinitely small superficial element of area at a point outside the curve, and t, t' the lengths of the tangents from the point to the curve, and θ the angle of intersection of these tangents: prove that the sum of the quantities represented by $\frac{ds\sin\theta}{t\times t'}$ taken for all points exterior to the curve is equal to $2\pi^2$. (M. Moreau, Nouv. An. de Math. 1874).

13. If a curve be such that the area between any portion and a fixed right line is proportional to the corresponding length of the curve, show that it is a catenary.

14. Prove that the volume of a rectangular parallelopiped is to that of its least circumscribed ellipsoid as $2 : \pi\sqrt{3}$.

15. Prove that

$$\int_0^\alpha\frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}}=\int_0^\beta\frac{d\theta}{\sqrt{\kappa^2-\sin^2\theta}},$$

where $\sin\beta=\kappa\sin\alpha$.

16. If any number of triangles be inscribed in one ellipse, and circumscribed to another ellipse, concentric and similar, prove that these triangles have all the same area.

17. Show that the value of the integral

$$\int_a^b\frac{dy}{\sqrt{y^m-1}}$$

may be exhibited by the following geometrical construction. Let the curve whose equation is $r^{\frac{m}{m+2}}\cos\frac{m}{m+2}\omega=1$ roll on the axis of x, take the points $(x_1, y_1,)$ (x_2, y_2) on the roulette described by the pole, such that $y_1=a$, $y_2=b$, then

$$\int_a^b\frac{dy}{\sqrt{y^m-1}}=x_2-x_1. \qquad \text{(Mr. Jellett.)}$$

18. If s be the length of the arc of a spherical curve measured to any point P, and t be the intercept on the great circle touching at P, between the point of contact and the foot of the perpendicular from the pole, prove that

$$s-t=\int\sin p\,d\omega.$$

The proof is similar to that of the corresponding theorem in plano. See Art. 154.

19. Prove that the volume of a polyhedron, having for bases any two polygons situated in parallel planes, and for lateral faces trapeziums, is expressed by the formula

$$\frac{H}{6}(B + B' + 4B'');$$

where H is the distance between the parallel planes, B and B' the areas of the polygonal bases, and B'' the area of the section equidistant from the two bases.

20. If S be the length of a loop of the curve $r^n = a^n \cos n\theta$, and A the area of a loop of the curve $r^{2n} = a^{2n} \cos 2n\theta$, prove that

$$A \times S = \frac{\pi a^3}{2n}.$$

21. Find approximately the area, and also the length, of a loop of the curve $r^{\frac{5}{3}} = a^{\frac{5}{3}} \cos \frac{5\theta}{3}$; (see Dif. Cal. Art. 261).

Ans. area $= a^2 \times 0.56616$,
length $= a \times 2.72638$.

22. Show from Art. 134, that if a parabola roll on a right line, the locus of its focus is a catenary.

23. If A be the area of any oval, B that of its pedal with respect to any internal origin O, and C that of the locus of the point on the perpendicular whose distance from O is equal to distance of point of contact from O; prove that A, B, C, are in arithmetical progression.

24. The arc of a curve is connected with the abscissa by the equation $s^2 = kx$, find the curve.

25. If the co-ordinates of a point on a curve be given by the equations,

$$x = c \sin 2\theta\,(1 + \cos 2\theta), \quad y = c \cos 2\theta\,(1 - \cos 2\theta),$$

prove that the length of its arc, measured from the origin, is

$$\frac{4}{3} c \sin 3\theta.$$

26. Show how to find the sum of every element of the periphery of an ellipse divided by any odd power $(2r + 1)$ of the semi-diameter conjugate to that which passes through the element, and give the result in the case of the fifth power. (Mr. W. Roberts.)

Ans. $\dfrac{4}{(ab)^{2r-1}} \displaystyle\int_0^{\frac{\pi}{2}} (a^2 \cos^2\theta + b^2 \sin^2\theta)^{r-1}\, d\theta.$

This gives $\dfrac{\pi(a^2 + b^2)}{a^3 b^3}$ when $r = 2$.

27. A sphere intersects a right cylinder, prove that the entire surface of the cylinder included within the sphere is equal to the product of the diameter of the cylinder into the perimeter of an ellipse, whose axes are equal to the greatest and least intercepts made by the sphere on the edges of the cylinder.

28. Show that the equations of the involute of a circle are of the form

$$x = a\cos\phi + a\phi\sin\phi,\quad y = a\sin\phi - a\phi\cos\phi,$$

and prove that the length of the arc of this involute, measured from $\phi = 0$, is one-half of the arc of a circle which would be described by a radius equal to the arc of its evolute moving through the angle ϕ.

29. Show that the area of the cassinoid

$$r^4 - 2a^2r^2\cos 2\theta + a^4 = b^4$$

is expressed by aid of an elliptic arc when $b > a$; and by an hyperbolic arc when $a > b$.

30. A string AB, of given length, lies in contact with a plane convex curve with its end A fixed; the string is unwound, and B is made to move about A till the string is again wound on the curve, the final position of B being B'; prove that for variations of the position of A the arc traced out by B will be a maximum or a minimum, when the tangents at B and B' are equally inclined to the tangent at A; and will be the former or the latter, according as the curvature at A is greater or less than half the sum of the curvatures at B and B'.

31. Find the value of

$$\int_0^\infty e^{-\left(\frac{a}{x} + \beta x\right)} \frac{dx}{\sqrt{x}}. \qquad \textit{Ans.}\ \sqrt{\frac{\pi}{\beta}}\, e^{-2\sqrt{a\beta}}.$$

32. Find the length and also the area of the pedal of a cissoid, the vertex being origin.

$$\textit{Ans.}\ \frac{8a}{\sqrt{3}}\log(2+\sqrt{3}) - 4a;\ \frac{\pi a^2}{24}.$$

33. Prove that the length of an arc of the lemniscate $r^2 = a^2\cos 2\theta$, is represented by the integral

$$\frac{a}{\sqrt{2}}\int \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}.$$

34. Integrate the equation

$$\cos\theta\,(\cos\theta - \sin a\sin\phi)\,d\theta + \cos\phi\,(\cos\phi - \sin a\sin\theta)\,d\phi = 0.$$

If the arbitrary constant be determined by the condition that the equation must be satisfied by the values $(0, a)$ of (θ, ϕ), show that the equation is satisfied by putting $\theta + \phi = a$.

35. Each element of the surface of an ellipsoid is divided by the area of the parallel central section of the surface, find the sum of all the elementary quotients extended through the entire ellipsoid. *Ans.* 4.

36. Hence, show that

$$\int_0^h\int_h^k \frac{(\mu^2 - \nu^2)\,d\mu\,d\nu}{\sqrt{\mu^2 - h^2}\,\sqrt{k^2 - \mu^2}\,\sqrt{h^2 - \nu^2}\,\sqrt{k^2 - \nu^2}} = \frac{\pi}{2}.$$

This depends on the expression for an element of the surface of an ellipsoid in terms of the elliptic co-ordinates of a point. See Salmon's Surfaces, Art. 426. This proof is due to M. Chasles (Liouville, tome III., p. 10).

37. Hence prove the relation

$$F(m)\ E(n) + F(n)\ E(m) - F(n)\ F(m) = \frac{\pi}{2},$$

where

$$F(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m^2 \sin^2\theta}}, \quad E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1 - m^2 \sin^2\theta}\, d\theta,$$

and $m^2 + n^2 = 1$.

Let $\nu = h \sin\theta$, and $\mu = \sqrt{h^2 \sin^2\phi + k^2 \cos^2\phi}$, in the preceding, and it becomes

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{h^2 \sin^2\phi + k^2 \cos^2\phi - h^2 \sin^2\theta}{\sqrt{h^2 \sin^2\phi + k^2 \cos^2\phi}\ \sqrt{k^2 - h^2 \sin\theta}}\, d\theta d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sqrt{h^2 \sin^2\phi + k^2 \cos^2\phi}}{\sqrt{k^2 - h^2 \sin^2\theta}}\, d\theta d\phi + \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sqrt{k^2 - h^2 \sin^2\theta}}{\sqrt{h^2 \sin^2\phi + k^2 \cos^2\phi}}\, d\theta d\phi$$

$$- \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{k^2\, d\theta d\phi}{\sqrt{h^2 \sin^2\phi + k^2 \cos^2\phi} \cdot \sqrt{k^2 - h^2 \sin^2\theta}}.$$

This furnishes the required result on making $h = mk$.

The preceding formula, which is due to Legendre, gives a general relation between complete elliptic functions of the first and second species, with complementary moduli.

38. If three curves be described on the surface of an ellipsoid, along the first of which the perpendicular to the tangent plane makes the angle γ with the axis of z, along the second β with the axis of y, and along the third α with the axis of x, and if the angles be connected by the relations $\frac{\tan\alpha}{a} = \frac{\tan\beta}{b} = \frac{\tan\gamma}{c}$; then, if A_3, A_2, A_1, be the included portions of the ellipsoidal surface, prove that

$$\frac{A_3 - A_2}{a^2} + \frac{A_1 - A_3}{b^2} + \frac{A_2 - A_1}{c^2} = 0. \qquad \text{(Mr. Jellett.)}$$

39. Show that the results in Arts. 158 and 159 are true for spherical conics where the tangents are arcs of great circles on the sphere.

40. Find an expression for the area of any plane section of an elliptic paraboloid; and show that the volume of the intercepted portion of the paraboloid is half that of a cylinder standing on the section and terminated by the parallel tangent plane.

Here (Art. 179), the section of an ellipsoid is

$$\pi a'b' \left(\frac{2h}{c'} - \frac{h^2}{c'^2}\right) = \frac{\pi abc}{\sin\omega\, c'} \left(\frac{2h}{c'} - \frac{h^2}{c'^2}\right).$$

On substituting $a^2 = pc$, $b^2 = qc$, $\frac{c}{c'} = 1$, and $c = \infty$, this becomes for the paraboloid,

$$\frac{2\pi\sqrt{pq}}{\sin\omega}\, h.$$

Again, the volume of the circumscribing cylinder is got by multiplying this area by $h\sin\omega$, and, consequently, is

$$2\pi\sqrt{pq}\, h^2.$$

41. Prove that the volume of the segment cut from a paraboloid by any plane is $\frac{3}{2}$ths of that of the circumscribing cone standing on the section made by the plane as base.

42. Prove that the value of

$$\int_0^\infty \frac{\cos bx \sin ax}{x}\, dx \text{ is } 0,\ \frac{\pi}{4},\ \text{or } \frac{\pi}{2},$$

according as b is $>$, $=$, or $< a$.

43. Prove that

$$\int_0^\infty \frac{\sin bx \sin ax}{x^2}\, dx$$

$= \frac{\pi}{2}$ multiplied by the lesser of the numbers a and b.

44. If e be the eccentricity of an ellipse whose semiaxis major is unity, and E the length of its quadrant, prove that

$$\int_0^h \frac{E e\, de}{(1 - e^2)\sqrt{h^2 - e^2}} = \frac{\pi h}{2\sqrt{1 - h^2}}. \qquad \text{(Mr. W. Roberts.)}$$

45. If S represent the length of a quadrant of the curve $r^m = a^m \cos m\theta$, and S_1 the quadrant of its first pedal, prove that

$$S \,.\, S_1 = \frac{m+1}{2m}\, \pi a^2.$$

Here (Ex. 3, p. 206), we have

$$S = \frac{a\sqrt{\pi}}{2m} \frac{\Gamma\left(\frac{1}{2m}\right)}{\Gamma\left(\frac{m+1}{2m}\right)}.$$

Also, the first pedal (Dif. Cal. Art. 261) is derived by substituting $\frac{m}{m+1}$ instead of m, hence

$$S_1 = \frac{(m+1)a\sqrt{\pi}}{2m} \frac{\Gamma\left(\frac{m+1}{2m}\right)}{\Gamma\left(1+\frac{1}{2m}\right)};$$

$$\therefore \quad SS_1 = \frac{(m+1)\,\pi a^2}{4m^2} \frac{\Gamma\left(\frac{1}{2m}\right)}{\Gamma\left(1+\frac{1}{2m}\right)} = \frac{(m+1)\,\pi a^2}{2m}.$$

46. In general, if S_n be the quadrant of the n^{th} pedal of the curve in the last, prove that

$$S_{n-1}\, S_n = \frac{mn+1}{2m}\,\pi a^2.$$

Here, it is readily seen that the n^{th} pedal is got by substituting $\frac{m}{mn+1}$ instead of m in the equation of the proposed, ∴ &c. (Mr. W. Roberts, Liou. x., p. 177.)

THE END.